

TWO MINIMIZATION PROBLEMS ON NON-SCATTERING SOLUTIONS TO MASS-SUBCRITICAL NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. In this paper, we introduce two minimization problems on non-scattering solutions to nonlinear Schrödinger equation. One gives us a sharp scattering criterion, the other is concerned with minimal size of blowup profiles. We first reformulate several previous results in terms of these two minimizations. Then, the main result of the paper is existence of minimizers to the both minimization problems for mass-subcritical nonlinear Schrödinger equations. To consider the latter minimization, we consider the equation in a Fourier transform of generalized Morrey space. It turns out that the minimizer to the latter problem possesses a compactness property, which is so-called almost periodicity modulo symmetry.

1. INTRODUCTION

In this paper, we consider time global behavior of solutions to the following focusing nonlinear Schrödinger equation

$$(1.1) \quad \begin{cases} i\partial_t u + \Delta u = -|u|^{2\alpha} u, & (t, x) \in I \times \mathbb{R}^d \\ u(t_0, x) = u_0(x), \end{cases}$$

where $d \geq 1$, $I \subset \mathbb{R}$ is an interval, $t_0 \in I$, and $u(t, x) : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ is an unknown.

We introduce two minimization problems associated with time global behavior of solutions to (1.1). First, we briefly recall several previous results in terms of the two problems. Then, we consider some mass-subcritical cases $\alpha < 2/d$ and establish existence of minimizers to the both problems, which is the main result.

1.1. Two minimization problems. To be more precise, let us make some notation. Let X be a Banach space which corresponds to a *state space*. Suppose that X is so that $e^{it\Delta}$ becomes a linear bounded operator on X and that $e^{it\Delta}$ converges strongly to the identity on X as $|t| \rightarrow 0$. We also suppose that (1.1) is locally well-posed in X in the following sense: For any given $u_0 \in X$ and $t_0 \in \mathbb{R}$, there exist an interval $I \ni t_0$ and a unique function $u(t, x) \in I \times \mathbb{R}^d \rightarrow \mathbb{C}$ such that $u(t) \in C(I, X)$ and the equality

$$(1.2) \quad u(t) = e^{i(t-t_0)\Delta} u_0 + i \int_{t_0}^t e^{i(t-s)\Delta} (|u|^{2\alpha} u)(s) ds \quad \text{in } X$$

holds for all $t \in I$. Moreover, continuous dependence on the data holds: If $u_{0,n} \in X$ converges to u_0 in X as $n \rightarrow \infty$ then a sequence of corresponding solutions $u_n(t)$ with data $u_n(t_0) = u_{0,n}$ converges to $u(t)$ in $L^\infty(J, X)$ as

$n \rightarrow \infty$ for any $J \subset\subset I$. We say that u is a maximal-lifespan solution if it cannot be extended to any strictly larger interval. Let $I_{\max} = I_{\max}(u) := (T_{\min}(u), T_{\max}(u))$ be the maximal interval of a u .

We say a solution $u(t)$ *scatters* in X for positive time direction (resp. negative time direction) if $T_{\max} = +\infty$ (resp. $T_{\min} = -\infty$) and if $e^{-it\Delta}u(t)$ has a strong limit in X as $t \rightarrow \infty$ (resp. $t \rightarrow -\infty$).

First minimization problem. We now introduce the first problem. Let $\ell : X \rightarrow \mathbb{R}_{\geq 0}$ be a *size function*. Suppose that ℓ is continuous. The first minimization is the following;

$$E_1 = E_1(\alpha, X, \ell) \\ := \inf \left\{ \inf_{t \in I_{\max}(u)} \ell(u(t)) \mid \begin{array}{l} u(t) : \text{solution to (1.1) that does not} \\ \text{scatter for positive time direction.} \end{array} \right\}$$

When $\ell(\cdot) = \|\cdot\|_X$, we simply denote $E_1(\alpha, X)$. If a size function $\ell(\cdot)$ is invariant under complex conjugation, that is, if

$$(1.3) \quad \ell(\bar{f}) = \ell(f), \quad \forall f \in X$$

is satisfied then it is obvious from time symmetry of (1.1) that

$$E_1 = \inf \left\{ \inf_{t \in I_{\max}(u)} \ell(u(t)) \mid \begin{array}{l} u(t) : \text{solution to (1.1) that does not} \\ \text{scatter for negative time direction.} \end{array} \right\} \\ = \inf \left\{ \inf_{t \in I_{\max}(u)} \ell(u(t)) \mid \begin{array}{l} u(t) : \text{solution to (1.1) that does not} \\ \text{scatter for at least one time direction.} \end{array} \right\}.$$

Notice that the validity of small data scattering result in X is expressed as $E_1(\alpha, X, \ell) > 0$. In other words, the positivity of E_1 suggests that (X, ℓ) is a suitable framework to consider time global behavior of solutions. The value E_1 gives us a sharp scattering criterion;

$$\ell(u_0) < E_1 \implies u(t) \text{ scatters for positive time direction.}$$

Further, $u(t)$ scatters for both time direction if ℓ satisfies (1.3).

Also remark that $E_1(\alpha, X) < \infty$ is equivalent to existence of a nonscattering solution in X .

Second minimization problem. The next problem is

$$E_2 = E_2(\alpha, X, \ell) \\ := \inf \left\{ \limsup_{t \uparrow T_{\max}(u)} \ell(u(t)) \mid \begin{array}{l} u(t) : \text{solution to (1.1) that does not} \\ \text{scatter for positive time direction.} \end{array} \right\}.$$

As in E_1 , a similar infimum value for negative time direction has the same value under the assumption (1.3). Intuitively, E_2 is a minimum size of possible “blowup profiles.”¹

¹ It is known that, in some cases, a solution that does not scatter for positive time direction tends to an orbit of a static profile, a blowup profile, by a group action, say G , as time approaches to the end of maximal time interval (e.g., a standing wave solution $u(t, x) = e^{it\omega} \phi_\omega(x)$). If a size function ℓ is chosen so that it is invariant under the group action G , then the size of such a solution tends to that of a corresponding profile. Of course, another kind of behavior may take place, in general, and so it may not be a definition of E_2 .

It is obvious by definition that

$$0 \leq E_1 \leq E_2 \leq \infty.$$

Further, if $\ell(\cdot)$ is a time independent quantity, such as L^2 -norm of the solution, then two infimum values coincide. On the other hand, it may happen that $E_1 < E_2$ (see Theorems 1.2 and 1.6). A good point on these minimizations is that $\ell(\cdot)$ needs not to be a time independent quantity. Hence, it enables us to consider the problem under a setting that no conserved quantity is available.

Besides $E_1 = \infty$ implies non-existence of a nonscattering solution, $E_2 = \infty$ gives us a weaker statement that any bounded (in the sense of a corresponding size function) solution scatters. This kind of scattering result is extensively studied in the defocusing case (see [23, 25, 36, 38, 39, 40], for instance). Although we need some modification on E_1 and E_2 , there is an example of the setting that yields $E_1 < E_2 = \infty$ (see Theorems 1.6 and 1.7).

In this way, we can obtain somewhat detailed information on dynamics from a combination of these two values.

1.2. Evaluation of E_1 and E_2 for mass-critical and -supercritical cases. The main interest here is to find explicit values of E_1 and E_2 . Further, we seek minimizers. Besides its own interest, to know a minimizer would be a key step of finding the explicit infimum values. In some mass-critical and -supercritical settings, a *ground state solution* $Q_\alpha(t, x)$ attains E_1 and/or E_2 , where $Q_\alpha(t, x)$ is given by

$$Q_\alpha(t, x) := \begin{cases} e^{it}Q(x), & 0 < \alpha < \frac{2}{d-2}, \\ W(x) = \left(1 + \frac{|x|^2}{d(d-2)}\right)^{-\frac{d-2}{2}}, & \alpha = \frac{2}{d-2} \end{cases}$$

Here, $Q(x)$ is a positive radial solution of $-\Delta Q + Q = Q^{2\alpha+1}$ and W solves $-\Delta W = W^{\frac{d+2}{d-2}}$.

Let us now collect several settings that explicit value of E_1 and E_2 can be determined. They are reformulations of previous results. Let us begin with the mass-critical case.

Theorem 1.1 (mass-critical case). *Let $d \geq 1$ and $\alpha = 2/d$. Let $X = L^2(\mathbb{R}^d)$ and $\ell(\cdot) = \|\cdot\|_{L^2}$. Then, $E_1 = E_2 = \|Q\|_{L^2(\mathbb{R}^d)}$.*

This result is just a rephrase of Dodson [8] (see also [24, 27]). The next case is energy-critical case.

Theorem 1.2 (energy-critical case). *Let $d \geq 4$ and $\alpha = 2/(d-2)$. Let $X = \dot{H}^1(\mathbb{R}^d)$ and $\ell(\cdot) = \|\cdot\|_{\dot{H}^1}$. Then,*

$$E_1 = \sqrt{\frac{2}{d}} \|W\|_{\dot{H}^1(\mathbb{R}^d)}, \quad E_2 = \|W\|_{\dot{H}^1(\mathbb{R}^d)}$$

Moreover, there exists a global radial solution $W_-(t)$ such that

- (1) $W_-(t)$ scatters for negative time direction and $\lim_{t \rightarrow -\infty} \|W_-(t)\|_{\dot{H}^1(\mathbb{R}^d)} = E_1$;

- (2) $W_-(t)$ converges to W exponentially as $t \rightarrow \infty$, that is, there exists positive constants c and C such that

$$\|W_-(t) - W\|_{\dot{H}^1(\mathbb{R}^d)} \leq Ce^{-ct}$$

for all $t \geq 0$. In particular, $\lim_{t \rightarrow \infty} \|W_-(t)\|_{\dot{H}^1(\mathbb{R}^d)} = E_2$.

Furthermore, there is no solution $u(t)$ which does not scatter for positive time direction and which attains E_1 at some finite time. That is, if $\|u(t_0)\|_{\dot{H}^1} = E_1$ for some $t_0 \in I_{\max}(u)$ then $u(t)$ scatters for both time directions.

The above theorem follows by summarizing several previous results [7, 10, 22, 26, 30]. We give a proof in Appendix A.

Remark 1.3. In the 3d cubic case ($d = 3$ and $\alpha = 1$, $\dot{H}^{1/2}$ -critical), characterizations for $E_1(1, H^1(\mathbb{R}^3), \ell)$ and $E_2(1, H^1(\mathbb{R}^3), \ell)$ similar to Theorem 1.2 can be obtained from results in [1, 9, 11, 19], where $\ell(f) := \|f\|_{L^2}^{1/2} \|\nabla f\|_{L^2}^{1/2}$. It would be possible to extend the result to whole inter-critical cases, i.e., between mass-critical and energy-critical case. However, we do not pursue it any more.

1.3. Mass-subcritical case and weighted L^2 spaces. Now, we turn to the mass-subcritical case $\alpha < 2/d$, which is the main interest of this paper. It is well-known that global well-posedness holds in L^2 or H^1 . These are natural spaces in which the conserved quantities make sense. However, these spaces are not suitable to consider time global behavior because a scaling argument shows that $E_2(\alpha, L^2) = E_2(\alpha, H^1) = 0$. Then, we need some other space X to have $E_1 > 0$. The main purpose of the paper is to see that a hat-Morrey space is a good candidate for this kind of analysis.

Before this, we briefly recall some previous results in weighted L^2 spaces. Weighted L^2 spaces are frequently used for the analysis of the mass-subcritical case and are spaces in which small data scattering holds. In this paper, we consider the following two weighted spaces

$$\mathcal{F}H^1 := \{f \in \mathcal{S}' \mid \mathcal{F}f \in H^1\}, \quad \mathcal{F}\dot{H}^s := \{f \in \mathcal{S}' \mid \mathcal{F}f \in L^{\frac{2d}{d-2s}} \cap \dot{H}^s\},$$

where $0 < s < d/2$ and \mathcal{F} stands for the Fourier transform with respect to the space variable.

To consider (1.1) in a weighted L^2 space, we generalize the concept of solution by introducing an *interaction variable* $v(t) := e^{-it\Delta}u(t)$. Let $X = \mathcal{F}H^1$ or $X = \mathcal{F}\dot{H}^s$. Notice that $e^{it\Delta}$ ($t \neq 0$) is not a bounded operator on X any more. For given u_0 such that $v_0 = e^{-it_0\Delta}u_0 \in X$, we say $u(t, x) \in I \times \mathbb{R}^d \rightarrow \mathbb{C}$ is a solution to (1.1) on an interval $I \ni t_0$ if $v(t)$ belongs to $C(I, X)$ and satisfies

$$v(t) = v_0 + i \int_{t_0}^t e^{-is\Delta} [|e^{is\Delta}v(s)|^{2\alpha} (e^{is\Delta}v(s))] ds \quad \text{in } X$$

for all $t \in I$. The continuous dependence is also defined as a continuity of $v_0 \mapsto v(t)$ as in the previous case. It is worth mentioning that it may happen that $u(0) \in X$ but $u(t) \notin X$ for all $t \neq 0$. In particular, this implies that a time translation symmetry breaks down.

Remark 1.4. Remark that if X is so that $e^{it\Delta}$ becomes a linear bounded operator on X and that $e^{it\Delta}$ converges strongly to identity on X as $|t| \rightarrow 0$, then modified notion of well-posedness coincides with the original one. Indeed, $v(t) = e^{-it\Delta}u(t) \in C(I, X)$ is equivalent to $u(t) \in C(I, X)$ and the above modified integral formula is equivalent to (1.2). In this sense, this modified formulation is a generalization.

We then modify the definition of E_1 slightly; fix $t_0 = 0$ and

$$\tilde{E}_1(\alpha, X, \ell) := \inf \left\{ \ell(u(0)) \mid \begin{array}{l} u(t) : \text{solution to (1.1) that does not} \\ \text{scatter for positive time direction.} \end{array} \right\}.$$

Namely, we measure size of solutions at a specific time because time translation symmetry is broken. Similarly, the infimum E_2 associated with the weighted L^2 spaces are defined in a modified form

$$\tilde{E}_2 = \inf \left\{ \limsup_{t \uparrow T_{\max}(u)} \ell(e^{-it\Delta}u(t)) \mid \begin{array}{l} u(t) : \text{solution to (1.1) that does not} \\ \text{scatter for positive time direction.} \end{array} \right\}.$$

Remark 1.5. By Remark 1.4, we can introduce \tilde{E}_1 and \tilde{E}_2 under the original definition of well-posedness. In that case, we have $\tilde{E}_1 = E_1$ thanks to the time translation symmetry. Further, if ℓ is invariant under $e^{it\Delta}$ then $\tilde{E}_2 = E_2$.

It is known that the weighted spaces are the spaces in which small data scattering holds, i.e., $\tilde{E}_1(\alpha, X) > 0$ (see [13, 41, 34] and references therein). Further, in [33, 34], existence of a minimizer to \tilde{E}_1 is shown for suitable size functions. However, it will turn out soon that \tilde{E}_2 is not finite. It would suggest that the weighted L^2 spaces are not so good frameworks for the second minimization.

Let us now introduce precise results in the weighted L^2 framework. Let $s_c := d/2 - 1/\alpha$. Remark that $s_c < 0$ if and only if $\alpha < 2/d$.

Theorem 1.6 (mass-subcritical case I, [33, 34]). *Let $d \geq 1$ and $\max(1/d, 2/(d+2)) < \alpha < 2/d$. Let $X = \mathcal{F}H^1$ and*

$$\ell(f) = \ell_{\mathcal{F}H^1}(f) := \| |x|f \|_{L^2}^{-s_c} \|f\|_{L^2}^{1+s_c}.$$

Then, $0 < \tilde{E}_1 < \ell_{\mathcal{F}H^1}(Q) < \tilde{E}_2 = \infty$. Further, there exists a solution $u_c(t)$ that attains \tilde{E}_1 , i.e., $\ell_{\mathcal{F}H^1}(u_c(0)) = \tilde{E}_1$.

Notice that $\tilde{E}_2 = \infty$ implies that even a minimizer $u_c(t)$ to \tilde{E}_1 satisfies $\sup_{t \geq 0} \|e^{-it\Delta}u_c(t)\|_{\mathcal{F}H^1} = \infty$. The proof of $\tilde{E}_2 = \infty$ is immediate from time decay property that the boundedness of $\ell_{\mathcal{F}H^1}(e^{-it\Delta}u(t))$ gives. The infiniteness of \tilde{E}_2 can be understood also as a reflection of the fact that $\ell_{\mathcal{F}H^1}(e^{-it\Delta}u(t))$ is a scattering-solution-oriented value. For example, even the ground state solution is not bounded;

$$\sup_{t \in \mathbb{R}} \ell_{\mathcal{F}H^1}(e^{-it\Delta}Q_\alpha(t)) = \infty.$$

Intuitively, if $u(t)$ does not scatter then $e^{-it\Delta}$ gives some uncanceled “dispersion effect,” which penalized by the weighted L^2 norm.

We have a similar result in the case $X = \mathcal{F}\dot{H}^{|s_c|}$.

Theorem 1.7 (mass-subcritical case II, [34]). *Let $d \geq 1$ and $\max(1/d, 2/(d+2)) < \alpha < 2/d$. Let $X = \mathcal{F}\dot{H}^{|s_c|}$ and $\ell(\cdot) = \|\cdot\|_{\mathcal{F}\dot{H}^{|s_c|}}$. Then, $0 < \tilde{E}_1 < \ell(Q) < \tilde{E}_2 = \infty$. Further, there exists a solution $u_c(t)$ that attains \tilde{E}_1 , i.e., $\ell(u_c(0)) = \tilde{E}_1$.*

In [34], the identity $\tilde{E}_2 = \infty$ is not shown. A proof will appear elsewhere.

1.4. Mass-subcritical case and hat-Morrey space. Let us proceed to the main issue. In this paper, we want to consider (1.1) in mass-subcritical case with choosing a state space X and a size function ℓ so that the both E_1 and E_2 becomes positive and finite. As seen in the previous section, if we choose a weighted L^2 space as a state space, then the finiteness of E_2 problem is not clear. Hence, we will seek another space.

The conclusion is that a hat-Morrey space is a good candidate. The space, introduced by Bourgain [5], is used in a refinement of a Stein-Tomas estimate, a special case of Strichartz' estimate, see [4, 35, 6, 3]. The definition is as follows.

Definition 1.8. *For $j \in \mathbb{Z}$, we let*

$$\mathcal{D}_j := \left\{ 2^{-j}([0, 1]^d + k) = \prod_{l=1}^N [k_l 2^{-j}, (k_l + 1) 2^{-j}] \mid k \in \mathbb{Z}^d \right\}$$

be a set of dyadic cubes with size 2^{-j} . Let $\mathcal{D} := \cup_{j \in \mathbb{Z}} \mathcal{D}_j$. For a cube $\tau \in \mathcal{D}$, we frequently use the notation $\tau = \tau_k^j := 2^{-j}([0, 1]^d + k)$ with suitable $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. Let us introduce a (generalized) Morrey norm by

$$\|f\|_{M_{q,r}^p} := \left\| |\tau_k^j|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(\tau_k^j)} \right\|_{\ell^r_{j \in \mathbb{Z}, k \in \mathbb{Z}^d}}$$

for $1 \leq q \leq p \leq r \leq \infty$. If $r < \infty$ we assume $q < p < r$. A hat-Morrey norm is also introduced by

$$\|f\|_{\hat{M}_{q,r}^p} := \|\mathcal{F}f\|_{M_{q',r}^{p'}} = \left\| |\tau_k^j|^{\frac{1}{p'} - \frac{1}{q'}} \|\mathcal{F}f\|_{L^{q'}(\tau_k^j)} \right\|_{\ell^r_{j \in \mathbb{Z}, k \in \mathbb{Z}^d}}$$

for $1 \leq p \leq q \leq \infty$ and $p' \leq r \leq \infty$. Again, if $r < \infty$ we assume $q' < p' < r$. We define function spaces $M_{q,r}^p$ and $\hat{M}_{q,r}^p$ as sets of functions in $L_{\text{loc}}^q(\mathbb{R}^d)$ and $\mathcal{FL}_{\text{loc}}^{q'}(\mathbb{R}^d)$, respectively, such that the corresponding norm is finite.

Remark 1.9. We remark that $M_{q,\infty}^p$ coincide with usual Morrey and so that the $M_{q,r}^p$ is a generalization. If $r < \infty$ then $M_{p,r}^p = \hat{M}_{p,r}^p = \{0\}$. If $r < \infty$ then $M_{q,r}^r$ and $M_{q',r}^{r'}$ do not contain $\mathbf{1}_\tau(x)$ for any $\tau \in \mathcal{D}$, where $\mathbf{1}_A(x)$ denotes the characteristic function of a set $A \subset \mathbb{R}^d$.

The hat-Morrey space is a generalization of a hat-Lebesgue space $\hat{L}^p = \hat{M}_{p,\infty}^p = \mathcal{FL}^{p'}$. It is known that some dispersive estimates, such as the Strichartz' estimates, are naturally extended to the hat-Morrey and hat-Lebesgue spaces [4, 20, 32]. By means of these estimate, well-posedness of nonlinear Schrödinger equation is established in [15, 20, 32] for $d = 1$. In [14, 16, 31], well-posedness of KdV-type equations are studied in hat-Lebesgue spaces.

In [32], the first problem E_1 for generalized KdV equation is considered and they show that existence of a special solution which attains this value, in a suitable sense, under an assumption on a relation between the values of E_1 for generalized KdV equation and for nonlinear Schrödinger equation. As one tool for obtaining the result, well-posedness of (1.1) in a hat-Morrey space is established for $d = 1$. We first generalize this well-posedness for higher dimensions. Although our main interest is mass-subcritical case $\alpha < 2/d$, one does not need this “restriction” for a well-posedness result.

Theorem 1.10. *Let $d \geq 1$ and $\frac{2}{d} \cdot \frac{1}{1 + \frac{2}{d(d+3)}} < \alpha < \frac{2}{d} \cdot \frac{2}{1 - \frac{2}{d(d+3)}}$.*

- (1) *The equation (1.1) is locally well-posed in $\hat{L}^{d\alpha}$.*
- (2) *The equation (1.1) is locally well-posed in $\hat{M}_{q,r}^{d\alpha}$, provided*

$$d\alpha < q < \left(1 + \frac{2}{d(d+3)}\right) d\alpha,$$

and

$$(d\alpha)' < r \leq ((d+2)\alpha)^*,$$

where $a^* = \min(a, 2a/(a-2))$

Let us now turn to the minimization problems. For these problems, we assume that

$$(1.4) \quad \frac{2}{d} \cdot \frac{1}{1 + \frac{2}{d(d+3)}} < \alpha < \frac{2}{d}, \quad (d\alpha)' < r < ((d+2)\alpha)^*$$

and take $X = \hat{M}_{2,r}^{d\alpha}$ as a state space. Notice that the first assumption of (1.4) is necessary to take $q = 2$ in Theorem 1.10 (2), and the second assumption is exclusion of the end point $r = ((d+2)\alpha)^*$, which is necessary for concentration-compactness-type arguments. We remark that $\hat{M}_{2,r}^{d\alpha}$ is one of the spaces on which $\{e^{it\Delta}\}_t$ forms a one-parameter group of linear bijective isometries, and that $e^{it\Delta}$ converges strongly to the identity operator as $|t| \rightarrow 0$.

Introduce a size function as follows:

$$(1.5) \quad \ell_r(f) := \inf_{\xi \in \mathbb{R}^d} \left\| e^{-i(\cdot)\xi} f(\cdot) \right\|_{\hat{M}_{2,r}^{d\alpha}}.$$

One sees that ℓ_r is an equivalent quasi-norm on $\hat{M}_{2,r}^{d\alpha}$ (see Remark 3.2). Since ℓ_r satisfies (1.3), the meaning of E_1 and E_2 can be strengthened. An important fact is that the size of ground state is bounded in time

$$\ell_r(e^{it\Delta}Q) = \ell_r(Q) < \infty,$$

which gives us a desired a priori bound on the second minimization value,

$$E_2(\alpha, \hat{M}_{2,r}^{d\alpha}, \ell_r) \leq \ell_r(Q) < \infty.$$

Remark 1.11 (defocusing case). In this paper, we only consider focusing equations. However, the focusing nature is only used for the above a priori bound on E_2 . For the defocusing case, if we assume $E_2 < \infty$ then the same results as in Theorems 1.13 and 1.15 are obtained by the same proof. Further, if we obtain some contradiction from the conclusions in Theorems 1.13 and 1.15 then we have $E_2 = \infty$. It is needless to say that Theorem 1.10 holds without the boundedness assumption.

Now, let us introduce the main results of this paper.

Theorem 1.12. *Let $d \geq 1$ and suppose (1.4). Then, $0 < E_1(\alpha, \hat{M}_{2,r}^{d\alpha}, \ell_r) < \ell_r(Q)$.*

This theorem says two things. Firstly, E_1 is positive, that is, a small data scattering result holds in $\hat{M}_{2,r}^{d\alpha}$. Secondly, E_1 is strictly smaller than the size of ground state, and so the ground state solution is not a minimizer to the E_1 problem. As for a minimizer to E_1 , we have the following;

Theorem 1.13. *Let $d \geq 1$ and suppose (1.4). There exists a maximal-lifespan solution $u_{E_1}(t)$ to (1.1) such that*

- (1) $u_{E_1}(t)$ does not scatter for positive time direction.
- (2) $u_{E_1}(t)$ attains E_1 in such a sense that one of the following holds;
 - (a) $\ell(u_{E_1}(0)) = E_1$,
 - (b) $u_{E_1}(t)$ scatters for negative time direction and

$$\ell_r \left(\lim_{t \rightarrow -\infty} e^{-it\Delta} u_{E_1}(t) \right) = E_1.$$

Remark 1.14. In the above theorem, validity of the case (2)-(a) implies that existence of a minimizer as in Theorems 1.6 and 1.7. On the other hand, (2)-(b) corresponds to the situation as in the energy-critical case (Theorem 1.2). Notice that in the energy-critical case, the case (2)-(a) never happens.

We next state existence of a minimizer to E_2 of which flow is totally bounded modulo dilations, and translations in both physical and Fourier sides.

Theorem 1.15. *Let $d \geq 1$ and suppose (1.4). There exists a maximal-lifespan solution $u_{E_2}(t)$ to (1.1) such that*

- (1) $u_{E_2}(t)$ does not scatter for both time directions.
- (2) $\sup_{t \in I_{\max}(u_{E_2}) \cap \{t \geq 0\}} \ell_r(u_{E_2}(t)) = \sup_{t \in I_{\max}(u_{E_2}) \cap \{t \leq 0\}} \ell_r(u_{E_2}(t)) = E_2$.
- (3) $u_{E_2}(t)$ is almost periodic modulo symmetry i.e. there exist $y(t), z(t) : I_{\max}(u_{E_2}) \rightarrow \mathbb{R}^d$, $N(t) : I_{\max}(u_{E_2}) \rightarrow 2^{\mathbb{Z}}$, and $C(\eta) > 0$ such that

$$(1.6) \quad \sup_{|w| \leq \frac{N(t)}{C(\eta)}} \left\| (e^{iw \cdot (x-y(t))} - 1) u_{E_2}(t) \right\|_{\hat{M}_{2,r}^{d\alpha}} + \left\| \mathcal{F}^{-1} \mathbf{1}_{|\xi-z(t)| \geq C(\eta)N(t)} \mathcal{F} u_{E_2}(t) \right\|_{\hat{M}_{2,r}^{d\alpha}} \leq \eta$$

for any $\eta > 0$ and for any $t \in I_{\max}(u_{E_2})$.

Remark 1.16. The validity of (1.6) is equivalent to pre-compactness (or total boundedness) of the set

$$\left\{ \frac{e^{-iN(t)^{-1}x \cdot z(t)}}{N(t)^{1/\alpha}} u_{E_2} \left(t, \frac{x}{N(t)} + y(t) \right) \middle| t \in I_{\max} \right\} \subset \hat{M}_{2,r}^{d\alpha}$$

(see Theorem 2.19, below). $y(t)$ and $z(t)$ correspond to a spacial center and a frequency center, respectively. The meaning of the smallness of the first term in the left hand side of (1.6) is close to that of $\left\| \mathbf{1}_{|x-y(t)| \geq C(\eta)/N(t)} u_{E_2}(t) \right\|_{\hat{M}_{2,r}^{d\alpha}}$.

However, the equivalence of two smallnesses is not clear.

The ground state solution is an example that does not scatter and is almost periodic modulo symmetry. Namely, the ground state solution satisfies the first and third property of Theorem 1.15. In mass-critical and energy-critical cases, E_2 coincides with the size of ground state (Theorems 1.1 and 1.2). In the proofs of these theorems, a solution with almost periodicity modulo symmetry plays a crucial role. The main step of the proof there is to derive a contradiction from the assumption that E_2 is less than the size of ground state via a precise analysis on almost-periodic-modulo-symmetry solutions similar to that given in Theorem 1.15. In view of these facts, one conjecture in our case would be $E_2(\alpha, \hat{M}_{2,r}^{d\alpha}, \ell_r) = \ell_r(Q)$. This equality insists that every nonscatter solution $u(t)$ (even $u_{E_1}(t)$ given in Theorem 1.13) satisfies

$$\limsup_{t \uparrow T_{\max}} \ell_r(u(t)) \geq \ell_r(Q),$$

which seems reasonable from the view point of the soliton resolution conjecture. However, it is not clear even if we believe that there is no almost-periodic-modulo-symmetry solution “smaller” than the ground states, as in the mass-critical or energy-critical cases. One negative reason is that we do not know whether the size function ℓ_r is chosen well enough to capture such phenomena. An appropriate choice of a size function (for the above conclusion) would be given by a variational characterization of Q , which is not known in $\hat{M}_{2,r}^{d\alpha}$.

The rest of the paper is organized as follows. In Section 2, we introduce basic facts and several tools. In particular, Theorems 1.10 and 1.12 are established in this section. Section 3 is devoted to the study of a compactness result, a linear profile decomposition (Theorem 3.11). Then, we turn to the minimization problems. We prove Theorem 1.15 in Section 4, and Theorem 1.13 in Section 5.

2. PRELIMINARIES

2.1. Strichartz’ estimates. Strichartz’ estimate is a key tool for well-posedness theory. The estimates is naturally extended in terms of hat-Lebesgue spaces and hat-Morrey spaces. In one dimensional case, this kind of generalization is established in [20].

We first introduce Strichartz’ estimate in hat-Morrey space.

Proposition 2.1. *If*

$$\frac{1}{p} < \frac{d+1}{d+3} \min\left(\frac{1}{2}, \frac{1}{q}\right)$$

then

$$\|e^{it\Delta} f\|_{L_{t,x}^p(\mathbb{R}^{1+d})} \leq C \|f\|_{\hat{M}_{q,p^*}^{\frac{dp}{d+2}}},$$

where $p^* = \min(p, \frac{2p}{p-2})$.

The proof is similar to [3, Theorem 1.2] which corresponds to the case $p = 2(d+2)/d$. The condition of the proposition is necessary because the proof is based on the following bilinear restriction estimate.

Proposition 2.2 ([47]). *Let Q, Q' be cubes of sidelength 1 in \mathbb{R}^d such that*

$$\min\{|x - y| \mid x \in Q, y \in Q'\} \sim 1$$

and let f, g be functions such that \hat{f} and \hat{g} are supported in Q and Q' , respectively. Then, for all $p > \frac{2(d+3)}{d+1}$ and all q such that

$$\frac{1}{q} > \frac{d+3}{d+1} \frac{1}{p},$$

it holds that

$$\|(e^{it\Delta}f)(e^{it\Delta}g)\|_{L_{t,x}^{p/2}} \leq C \|f\|_{\hat{L}^q} \|g\|_{\hat{L}^q}$$

with a positive constant C independent of Q, Q', f , and g .

Let us proceed to Strichartz' estimates in hat-Lebesgue space. We have an embedding between hat-Morrey and hat-Lebesgue spaces.

Proposition 2.3. *We have the following embeddings.*

- *If $1 \leq q < p < r \leq \infty$ then $L^p \hookrightarrow M_{q,r}^p$.*
- *If $1 \leq q' < p' < r \leq \infty$ then $\hat{L}^p \hookrightarrow \hat{M}_{q,r}^p$.*

For the proof, see [4, 37, 3, 32]. The only one dimensional case is considered there, however the modification is obvious.

Proposition 2.4. *Let $d \geq 1$ and let $1 \leq p, q, r \leq \infty$ be such that*

$$\frac{2}{p} + \frac{d}{q} = \frac{d}{r}.$$

Assume that a triplet (p, q, d) is either $(p, q, d) = (\infty, \infty, 2)$, $(p, q, d) = (2, \frac{2d}{d-2}, d)$ with $d \geq 3$, or satisfies

$$0 \leq \frac{1}{q} \leq \frac{1}{2}, \quad 0 \leq \frac{1}{p} < \frac{1}{2} - \frac{1}{q}, \quad \frac{1}{p} \leq \frac{1}{4},$$

if $d = 1$;

$$0 < \frac{1}{q} \leq \frac{1}{2}, \quad 0 \leq \frac{1}{p} < \min\left(-\frac{2}{3q} + \frac{1}{2}, -\frac{3}{2q} + \frac{3}{4}\right)$$

if $d = 2$; and

$$\begin{aligned} 0 \leq \frac{1}{q} \leq \frac{1}{2}, \quad 0 \leq \frac{1}{p} \leq \frac{d}{d-2} \frac{1}{q}, \\ \frac{1}{p} < -\frac{d}{3} \left(\frac{1}{q} - \frac{d+1}{2(d+3)} \right) + \frac{d+1}{2(d+3)}, \\ \frac{1}{p} < -\frac{d+1}{2} \left(\frac{1}{q} - \frac{d+1}{2(d+3)} \right) + \frac{d+1}{2(d+3)} \end{aligned}$$

if $d \geq 3$. Then, it holds that

$$(2.1) \quad \|e^{it\Delta}g\|_{L_t^p L_x^q(\mathbb{R}^{1+d})} \leq C \|g\|_{\hat{L}^r(\mathbb{R}^d)}$$

and that

$$(2.2) \quad \left\| \int_{\mathbb{R}} e^{-it'\Delta} F(t') dt' \right\|_{\hat{L}^{r'}(\mathbb{R}^d)} \leq C \|F\|_{L_t^{p'} L_x^{q'}(\mathbb{R}^{1+d})}$$

for some positive constant C .

Proof. The second estimate (2.2) follows from (2.1) by duality. So, let us restrict our attention to the first estimate. The one dimensional case is due to Hyakuna and Tsutsumi [20]. Let us consider the multi dimensional case. The diagonal case $p = q > \frac{2(d+3)}{d+1}$ is an immediate consequence of Propositions 2.1 and 2.3. The off-diagonal case follows by interpolating the diagonal case and well-known $r = 2$ cases. \square

As for the Strichartz estimate for the hat-Lebesgue space, one can obtain a dual estimate (2.2). Note that a dual space of a hat-Morrey space is not clear; only a pre-dual space is characterized (see Section 2.3).

Let us now proceed to inhomogeneous estimates. For $t_0 \in \mathbb{R}$ and an interval $I \subset \mathbb{R}$ such that $I \ni t_0$, let

$$\Phi[F](t, x) = \int_{t_0}^t e^{i(t-t')\Delta} F(t') dt'.$$

The first estimate is as follows.

Corollary 2.5. *If*

$$1 \leq r < \frac{2}{1 - \frac{2}{d(d+3)}}$$

then

$$\|\Phi[F]\|_{L^\infty(I, \hat{L}^r)} \leq C \|F\|_{L_{t,x}^{\frac{(d+2)r}{2r+d}}(I \times \mathbb{R}^d)}$$

for any $F \in L_{t,x}^{\frac{(d+2)r}{2r+d}}(I \times \mathbb{R}^d)$.

This follows (2.2) and the fact that $e^{it\Delta}$ is an isometry on \hat{L}^r . The next one is an inhomogeneous estimate for non-admissible pairs.

Proposition 2.6 ([21, 12, 28, 49]). *Let* $1 \leq p_1, p_2, q_1, q_2 \leq \infty$. *The estimate*

$$\|\Phi[F]\|_{L_t^{p_1}(I, L_x^{q_1})} \leq C \|F\|_{L_t^{p'_2}(I, L_x^{q'_2})}$$

holds true if the following three assumptions are fulfilled:

- (acceptability) For $i = 1, 2$,

$$p_i < d \left(\frac{1}{2} - \frac{1}{q_i} \right) \quad \text{or} \quad (p_i, q_i) = (\infty, 2);$$

- (scale condition)

$$\sum_{i=1,2} \left(\frac{2}{p_i} + \frac{d}{q_i} \right) = d;$$

- (additional assumption) $q_1, q_2 < \infty$ if $d = 2$, and

$$(2.3) \quad \frac{1}{p_1} + \frac{1}{p_2} < 1, \quad \frac{d-2}{d} \leq \frac{q_1}{q_2} \leq \frac{d}{d-2}.$$

if $d \geq 3$

Further restricting to the case $p_1 = q_1$ and $p_2 = q_2$, we obtain

$$(2.4) \quad \|\Phi[F]\|_{L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)} \leq C \|F\|_{L_{t,x}^{\frac{(d+2)\alpha}{2\alpha+1}}(I \times \mathbb{R}^d)},$$

provided

$$(2.5) \quad \frac{2}{d} \cdot \frac{1}{1 + \frac{1}{d+1}} < \alpha < \frac{2}{d} \cdot \frac{1}{1 - \frac{1}{1+d}}.$$

This estimate is sufficient for our purpose. We remark that the condition (2.5) comes from the acceptability. It is known that the condition (2.3) can be relaxed slightly (see [12, 28, 49]). However, we do not recall it since, under the diagonal assumption, the condition (2.3) is already weaker than (2.5).

2.2. Well-posedness results. With the Strihartz' estimates, we obtain local well-posedness. The following norm plays an important role in the well-posedness theory.

Definition 2.7 (Scattering norm). *For an interval $I \subset \mathbb{R}$ and function $u(t, x) : I \times \mathbb{R}^d \rightarrow \mathbb{C}$, we define a scattering norm by*

$$S_I(u) := \|u\|_{L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)}.$$

Lemma 2.8. *Let $d \geq 1$ and $\alpha > 0$ satisfy (2.5). Then, there exists a constant $\delta = \delta(d) > 0$ such that if a function $u_0 \in \mathcal{S}'$ and an interval $I \subset \mathbb{R}$ satisfy $t_0 \in I$ and*

$$S_I(e^{i(t-t_0)\Delta} u_0) \leq \delta$$

then there exists a unique solution $u(t) : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ to (1.1) such that

$$S_I(u) \leq 2S_I(e^{i(t-t_0)\Delta} u_0).$$

Proof. By the non-admissible Strichartz' estimate (2.4),

$$\begin{aligned} \|\Phi[|u|^{2\alpha} u]\|_{L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)} &\leq C \| |u|^{2\alpha} u \|_{L_{t,x}^{\frac{(d+2)\alpha}{(2\alpha+1)}}(I \times \mathbb{R}^d)} \\ &= C \|u\|_{L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)}^{2\alpha+1} \end{aligned}$$

and similarly,

$$\begin{aligned} &\|\Phi[|u_1|^{2\alpha} u_1] - \Phi[|u_2|^{2\alpha} u_2]\|_{L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)} \\ &\leq C (\|u_1\|_{L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)} + \|u_2\|_{L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)})^{2\alpha} \|u_1 - u_2\|_{L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)} \end{aligned}$$

By a standard fixed point argument, we obtain a unique solution $u \in L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)$ of the equation $u(t) = e^{i(t-t_0)\Delta} u_0 + i\Phi[|u|^{2\alpha} u]$ if δ is sufficiently small. \square

Proof of Theorem 1.10. Let $t_0 = 0$ for simplicity. Notice that the assumption

$$\frac{2}{d} \cdot \frac{1}{1 + \frac{2}{d(d+3)}} < \alpha < \frac{2}{d} \cdot \frac{1}{1 - \frac{2}{d(d+3)}}$$

is stronger than (2.5).

Step 1. Since $\alpha > \frac{2}{d} \cdot \frac{1}{1 + \frac{2}{d(d+3)}}$, if $u_0 \in \hat{L}^{d\alpha}$ then

$$S_{\mathbb{R}}(e^{it\Delta} u_0) \leq C \|u_0\|_{\hat{L}^{d\alpha}}$$

by Proposition 2.4. The same conclusion is deduced from Proposition 2.1 for $u_0 \in \hat{M}_{q,r}^{d\alpha}$ if $\alpha > \frac{2}{d} \cdot \frac{1}{1 + \frac{2}{d(d+3)}}$, $q < (1 + \frac{2}{d(d+3)})d\alpha$, and $r \leq ((d+2)\alpha)^*$.

Step 2. Since $e^{it\Delta}u_0 \in L_{t,x}^{(d+2)\alpha}(\mathbb{R} \times \mathbb{R}^d)$, there exists an interval $I \ni 0$ such that

$$S_I(e^{it\Delta}u_0) \leq \delta,$$

where δ is the number given in Lemma 2.8. Then, the lemma gives a unique solution $u(t) \in L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)$.

Step 3. Let us show that $u(t)$ possesses the desired continuity. By Corollary 2.5, the Duhamel term obeys

$$\|\Phi[|u|^{2\alpha}u]\|_{L^\infty(I, \hat{L}^{d\alpha})} \leq C \|u\|_{L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)}^{2\alpha+1}$$

as long as $\alpha < \frac{2}{d} \cdot \frac{1}{1 - \frac{2}{d(d+3)}}$. Thus, $u(t) - e^{it\Delta}u_0 = i\Phi[|u|^{2\alpha}u] \in C(I, \hat{L}^{d\alpha})$.

Obviously, if $u_0 \in \hat{L}^{d\alpha}$ then the linear part satisfies $e^{it\Delta}u_0 \in C(I, \hat{L}^{d\alpha})$ and so $u(t) \in C(I, \hat{L}^{d\alpha})$. On the other hand, if $u_0 \in \hat{M}_{q,r}^{d\alpha}$ then $e^{it\Delta}u_0 \in C(I, \hat{M}_{q,r}^{d\alpha})$. Since $\hat{L}^{d\alpha} \hookrightarrow \hat{M}_{q,r}^{d\alpha}$ follows from the assumptions $q > d\alpha$ and $r > (d\alpha)'$, we conclude that $u(t) \in C(I, \hat{M}_{q,r}^{d\alpha})$.

Continuous dependence on the data follows by a standard argument. \square

Remark 2.9. Let us make a comment on the assumption of Theorem 1.10 on α . The lower bound of α is used in the existence part. Without this assumption, neither $u_0 \in \hat{L}^{d\alpha}$ nor $u_0 \in \hat{M}_{q,r}^{d\alpha}$ is sufficient to obtain a solution $u(t) \in L^{(d+2)\alpha}(I \times \mathbb{R}^d)$. On the other hand, the upper bound of α is used in the persistence-of-regularity part. Without this assumption, the obtained solution $u(t) \in L^{(d+2)\alpha}(I \times \mathbb{R}^d)$ does not necessarily belong to $C(I, \hat{L}^{d\alpha})$ or $C(I, \hat{M}_{q,r}^{d\alpha})$. Remark that one can obtain a solution under a weaker assumption (2.5) (and the lower bound on α).

Remark 2.10. If $u_0 \in \hat{L}^{d\alpha}$ then the solution $u(t)$ belongs to some off-diagonal space-time function space $L_t^p(I, L_x^q(\mathbb{R}^d))$, $p \neq q$. Indeed, $|u|^{2\alpha}u \in L_{t,x}^{\frac{(d+2)\alpha}{2\alpha+1}}(I \times \mathbb{R}^d)$ implies the Duhamel term $i\Phi[|u|^{2\alpha}u]$ belongs to $L_t^p(I, L_x^q(\mathbb{R}^d))$ for suitable (p, q) satisfying the assumption of non-admissible Strichartz's estimate (Proposition 2.6). Similarly, the linear part $e^{it\Delta}u_0$ also belongs to some $L_t^p(I, L_x^q(\mathbb{R}^d))$ by Proposition 2.4. Hence, the solution belongs to the intersection. A off-diagonal estimate, or a mixed-norm estimate, similar to Proposition 2.4 would be possible also for the hat-Morey space. However, we do not pursue it in this paper.

In the rest of this section, we let X be $\hat{L}^{d\alpha}$ or $\hat{M}_{q,r}^{d\alpha}$ that satisfies the assumption of Theorem 1.10. We next characterize finite time blowup and scattering in terms of the scattering norm of the solution. For the proof, see [31].

Proposition 2.11 (Blowup and scattering criterion). *Let $u(t)$ be an X -solution of (1.1) given in Theorem 1.10.*

- If $T_{\max} < \infty$ then $S_{[0,T)}(u) \rightarrow \infty$ as $T \uparrow T_{\max}$.
- The solution scatters for positive time direction if and only if $T_{\max} = +\infty$ and $S_{[0,\infty)}(u) < \infty$.

A similar statements are true for negative time direction.

An immediate consequence is small data scattering,

Theorem 2.12 (small data scattering). *Let $u(t)$ be a nonzero X -solution of (1.1) given in Theorem 1.10. If $S_{\mathbb{R}}(e^{i(t-t_0)\Delta}u_0) \leq \delta$ then $u(t)$ scatters in X for both time directions, where δ is the constant given in Lemma 2.8.*

The first part of Theorem 1.12 is a rephrase of this theorem. One has also a nonscattering result for solution with non-positive energy.

Theorem 2.13. *Let $u(t)$ be a nonzero X -solution of (1.1) given in Theorem 1.10. We further assume that $u_0 \in H^1$ and $E[u_0] := \frac{1}{2} \|\nabla u_0\|_{L^2}^2 - \frac{1}{p+1} \|u_0\|_{L^{2\alpha+1}}^{2\alpha+1} \leq 0$. Then, $u(t)$ is global and does not scatter for both time directions.*

The proof is similar to [31, Theorem 1.10]. Then, the rest of Theorem 1.12 is immediate by looking at energy of $cQ(x)$ for $0 < c \leq 1$ (see [32, 33, 34], for instance).

We also use the following stability estimate. For an interval $I \subset \mathbb{R}$, we say a function $u(t) \in C(I, X)$ is an X -solution to (1.1) on I with error $e(t) \in L_{t,x}^{(d+2)\alpha/(2\alpha+1)}(I \times \mathbb{R}^d)$ if $u(t)$ satisfies

$$u(t) = e^{i(t-t_0)\Delta}u(t_0) + i\Phi[|u|^{2\alpha}u](t) - i\Phi[e](t) \quad \text{in } X$$

for any $t, t_0 \in I$. Namely, $i\partial_t u + \Delta u = -|u|^{2\alpha}u + e$ (at least formally).

Proposition 2.14 (Long time stability). *Let $t_0 \in \mathbb{R}$ and $I \subset \mathbb{R}$ be a interval containing t_0 . Let $e(t), \tilde{e}(t) \in L_{t,x}^{(d+2)\alpha/(2\alpha+1)}(I \times \mathbb{R}^d)$. Let $\tilde{u} \in C(I, X)$ be an X -solution to (1.1) with error $\tilde{e}(t)$. Assume that \tilde{u} satisfies*

$$S_I(\tilde{u}) \leq M,$$

for some $M > 0$. Then there exists $\varepsilon_1 = \varepsilon_1(M)$ such that if $u(t_0), \tilde{u}(t_0) \in X$, $e(t)$, and $\tilde{e}(t)$ satisfy

$$S_I\left(e^{i(t-t_0)\Delta}(u(t_0) - \tilde{u}(t_0))\right) + \|e\|_{L_{t,x}^{\frac{(d+2)\alpha}{2\alpha+1}}(I \times \mathbb{R}^d)} + \|\tilde{e}\|_{L_{t,x}^{\frac{(d+2)\alpha}{2\alpha+1}}(I \times \mathbb{R}^d)} \leq \varepsilon$$

for some $0 < \varepsilon < \varepsilon_1$, then there exists an X -solution $u \in C(I, X)$ to (1.1) on the same I with error $e(t)$. Further, the following estimates are valid.

$$\begin{aligned} S_I(u - \tilde{u}) &\leq C\varepsilon, \\ \||u|^{2\alpha}u - |\tilde{u}|^{2\alpha}\tilde{u}\|_{L_{t,x}^{\frac{(d+2)\alpha}{2\alpha+1}}(I \times \mathbb{R}^d)} &\leq C\varepsilon, \\ \|u - \tilde{u}\|_{L^\infty(I, X)} &\leq \|u(t_0) - \tilde{u}(t_0)\|_X + C\varepsilon. \end{aligned}$$

The proof is standard. For instance, see [32, 33].

2.3. Functional analysis. In this section, we introduce two functional analysis results on Morrey spaces.

We first give a pre-dual of Morrey space $M_{q,r}^p$. This allows us to consider a weak-* convergence in Morrey and hat-Morrey spaces. In particular, thanks to the Banach-Alaoglu theorem, a closed unit ball of $M_{q,r}^p$ is compact with respect to weak-* topology. We use an argument similar to [17].

Definition 2.15. Let $1 \leq q \leq p \leq \infty$. We say a function g is a (p', q') -block with respect to a dyadic cube $Q \in \mathcal{D}$ if $\text{supp } g \subset \overline{Q}$ and

$$\|g\|_{L^{q'}(\mathbb{R}^d)} = \|g\|_{L^{q'}(Q)} \leq |Q|^{\frac{1}{q'} - \frac{1}{p'}}.$$

A function g is simply called a (p', q') -block if there exists a dyadic cube Q_0 such that g becomes a (p', q') -block with respect to Q_0 .

Definition 2.16. Let $1 \leq q \leq p \leq r \leq \infty$. The block space $N_{q', r'}^{p'}$ is the set of all measurable functions g for which there exists a decomposition

$$g(x) = \sum_{\mathbf{j} \in \mathbb{Z}^{1+d}} \lambda_{\mathbf{j}} A_{\mathbf{j}}(x),$$

where $A_{\mathbf{j}}$ is a (p', q') -block with respect to $\tau_{j'}^{j_1} = 2^{-j_1}([0, 1)^d + j')$ ($\mathbf{j} = (j_1, j') \in \mathbb{Z} \times \mathbb{Z}^d$), $\lambda_{\mathbf{j}} \in \ell_{\mathbf{j}}^{r'}(\mathbb{Z}^{1+d})$, and the convergence takes place for almost all $x \in \mathbb{R}^d$. The norm of g is given by

$$\|g\|_{N_{q', r'}^{p'}} = \inf \|\lambda_{\mathbf{j}}\|_{\ell_{\mathbf{j}}^{r'}(\mathbb{Z}^{1+d})},$$

where $\{\lambda_{\mathbf{j}}\}$ runs over all admissible expressions above.

Theorem 2.17. Let $1 \leq q \leq p \leq r \leq \infty$ be such that $M_{q, r}^p$ is defined and let $p > 1$. The generalized Morrey space $M_{q, r}^p$ is the dual of the block space $N_{q', r'}^{p'}$ in the following sense.

- (1) Let $f \in M_{q, r}^p$. Then, for any $g \in N_{q', r'}^{p'}$, we have $f \cdot g \in L^1(\mathbb{R}^d)$ and the mapping

$$N_{q', r'}^{p'} \ni g \mapsto \int_{\mathbb{R}^d} f(x)g(x)dx \in \mathbb{C}$$

defines a continuous linear functional L_f on $N_{q', r'}^{p'}$.

- (2) Conversely, any continuous linear functional L on $N_{q', r'}^{p'}$ can be realized as $L = L_f|_{N_{q', r'}^{p'}}$ with a certain $f \in M_{q, r}^p$. If $f_1, f_2 \in M_{q, r}^p$ define the same functional then $f_1 = f_2$ almost everywhere.

Furthermore, we have $\|L_f\|_{M_{q, r}^p \rightarrow \mathbb{C}} = \|f\|_{N_{q', r'}^{p'}}$.

Remark 2.18. By the above theorem, the dual of $\hat{M}_{q, r}^p$ is given by $\hat{N}_{q', r'}^{p'} = \{f \in \mathcal{S}' \mid \mathcal{F}f \in N_{q, r}^p\}$ with $(f, g)_{\hat{M}_{q, r}^p, \hat{N}_{q', r'}^{p'}} = \int_{\mathbb{R}^d} \mathcal{F}f(\xi) \mathcal{F}g(\xi) d\xi$, for $1 \leq q' \leq p' \leq r \leq \infty$ such that $\hat{M}_{q, r}^p$ is defined and $p < \infty$.

Proof. We first prove the first assertion. For given $g \in N_{q', r'}^{p'}$ and $\varepsilon > 0$, there exists a decomposition

$$g(x) = \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} \lambda_k^j g_k^j(x),$$

where each g_k^j is a (p', q') -block with respect to τ_k^j and

$$\|\lambda_k^j\|_{\ell_{j, k}^{r'}(\mathbb{Z}^{1+d})} \leq (1 + \varepsilon) \|g\|_{N_{q', r'}^{p'}}.$$

Then, twice use of Hölder's inequality yields

$$\begin{aligned}
\|fg\|_{L^1(\mathbb{R}^d)} &\leq \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} |\lambda_k^j| \int_{\tau_k^j} |f(x)g_k^j(x)| dx \\
&\leq \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} |\lambda_k^j| \|f\|_{L^q(\tau_k^j)} \|g_k^j\|_{L^{q'}} \\
&\leq \sum_{j \in \mathbb{Z}, k \in \mathbb{Z}^d} |\lambda_k^j| \left(|\tau_k^j|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(\tau_k^j)} \right) \\
&\leq \|f\|_{M_{q,r}^p} \left\| \lambda_k^j \right\|_{\ell_{j,k}^{r'}(\mathbb{Z}^{1+d})} \leq (1 + \varepsilon) \|f\|_{M_{q,r}^p} \|g\|_{N_{q',r'}^{p'}}.
\end{aligned}$$

Let us proceed to the second assertion. Let L be a bounded linear functional on $N_{q',r'}^{p'}$. For each $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$, the mapping

$$L^{p'}(\mathbb{R}^d) \ni g \mapsto L_k^j(g) \equiv L(g \mathbf{1}_{\tau_k^j}) \in \mathbb{C}$$

is a bounded linear functional. Since $p' \in [1, \infty)$, we see that L_k^j is realized by an $L_{\text{loc}}^p(\mathbb{R}^d)$ -function f_k^j with $\text{supp } f_k^j \subset \overline{\tau_k^j}$. Since $L_{k_1}^{j_1}(g) = L_{k_2}^{j_2}(g)$ holds by definition for all $g \in L^{p'}(\mathbb{R}^d)$ with $\text{supp } g \subset \tau_{k_1}^{j_1} \cap \tau_{k_2}^{j_2}$, one sees that there is a function $f \in L_{\text{loc}}^p(\mathbb{R}^d)$ such that $f_k^j(x) = f \mathbf{1}_{\tau_k^j}(x)$ for all $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. Then,

$$L_k^j(g) = \int_{\tau_k^j} g(x) f(x) dx$$

for all $g \in L_{\text{loc}}^{p'}(\mathbb{R}^d)$, $j \in \mathbb{Z}$, and $k \in \mathbb{Z}^d$. We now show that $f \in M_{q,r}^p$. We define

$$g_k^j(x) = \begin{cases} |\tau_k^j|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(\tau_k^j)}^{1-q} |f(x)|^{q-1} \mathbf{1}_{\tau_k^j}(x), & \|f\|_{L^q(\tau_k^j)} > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Remark that $q < \infty$ by the definition of $M_{q,r}^p$. Then, g_k^j is a (p', q') -block with respect to τ_k^j since $\|g_k^j\|_{L^{q'}} = |\tau_k^j|^{\frac{1}{p} - \frac{1}{q}}$. Fix a finite set $K \subset \mathbb{Z}^{1+d}$. Take an arbitrary nonnegative sequence $\{\rho_k^j\}_{j,k} \in \ell^{r'}(\mathbb{Z}^{1+d})$ supported on K and set

$$(2.6) \quad g_K = \sum_{(j,k) \in K} \rho_k^j g_k^j \in N_{q',r'}^{p'}.$$

We have

$$\sum_{(j,k) \in K} \rho_k^j |\tau_k^j|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(\tau_k^j)} = \int_{\mathbb{R}^d} |f(x)| g_K(x) dx = L(h),$$

where $h = g_K(x) \overline{\text{sgn}(f(x))}$. Further, by the decomposition (2.6),

$$|L(h)| \leq \|L\|_{N_{q',r'}^{p'} \rightarrow \mathbb{C}} \|h\|_{N_{q',r'}^{p'}} \leq \|L\|_{N_{q',r'}^{p'} \rightarrow \mathbb{C}} \left\| \rho_k^j \right\|_{\ell_{j,k}^{r'}(\mathbb{Z}^{1+d})}.$$

Since $r > 1$ and since K and $\{\rho_k^j\}$ are arbitrary, we conclude that

$$\|f\|_{M_{q,r}^p} \leq \|L\|_{N_{q',r'}^{p'} \rightarrow \mathbb{C}}.$$

Furthermore, $L = L_f |N_{q',r'}^{p'}|$ holds. The uniqueness follows by a standard argument. \square

We next characterize total boundedness of a bounded set $K \subset M_{q,r}^p$.

Theorem 2.19. *Let $1 \leq q < p < r < \infty$. A bounded set $K \subset M_{q,r}^p$ is totally bounded if and only if for any $\eta > 0$ there exists $C(\eta) > 0$ such that*

$$(2.7) \quad \|f \mathbf{1}_{\{|x| \geq C(\eta)\}}\|_{M_{q,r}^p} + \sup_{|z| \leq 1/C(\eta)} \|(T(z) - 1)f\|_{M_{q,r}^p} \leq \eta$$

for any $f \in K$, where $T(a)$ is a translation $(T(a)f)(x) = f(x - a)$, $a \in \mathbb{R}^d$.

Remark 2.20. This kind of characterization for L^p space for $1 \leq p < \infty$ is due to Kolmogorov, Tamarkin, Tulajkov, and Riesz [29, 43, 46, 48] (see also [18]). Further, when $p = 2$, a characterization in terms of Fourier transformation is given by Pego [42].

The proof relies on the following lemma.

Lemma 2.21 ([18]). *Let X be a metric space. Assume that, for every ε , there exist some δ , a metric space W , and a mapping $P : X \rightarrow W$ so that $P(X)$ is totally bounded, and if $x, y \in X$ are such that $d_W(P(x), P(y)) < \delta$ then $d_X(x, y) < \varepsilon$. Then X is totally bounded.*

Proof of Theorem 2.19. Let us first prove if a bounded set $K \subset M_{q,r}^p$ satisfies (2.7) then K is totally bounded.

Given $\varepsilon > 0$, take $C(\varepsilon) > 0$ so that (2.7) holds. Let $j_0 \in \mathbb{Z}$ be the maximum number that satisfies $2^{-j_0} > C(\varepsilon)$. Define $D_0 := [-2^{-j_0}, 2^{-j_0}]^d \subset \mathbb{R}^d$. Remark that $D_0 \supset \{|x| \leq C(\varepsilon)\}$. Let $j_1 = j_1(\varepsilon) \geq j_0$ be an integer to be chosen later. Recall the notation $\tau_k^j := 2^{-j}([0, 1]^d + k) \in \mathcal{D}_j$ with $j \in \mathbb{Z}$ and $k \in \mathbb{Z}^d$. Let $K_0 = \mathbb{Z}^d \cap [-2^{j_1-j_0}, 2^{j_1-j_0}]^d$ so that $D_0 = \bigcup_{k \in K_0} \tau_k^{j_1}$. We define a projection operator $\mathcal{P} = \mathcal{P}(\varepsilon) : M_{q,r}^p \rightarrow M_{q,r}^p$ as

$$\mathcal{P}f(x) := \begin{cases} \frac{1}{|\tau_k^{j_1}|} \int_{\tau_k^{j_1}} f(y) dy, & x \in \tau_k^{j_1}, \exists k \in K_0, \\ 0, & \text{otherwise.} \end{cases}$$

Remark that $\text{supp } \mathcal{P}f \subset \overline{D_0}$. \mathcal{P} is a bounded operator. Indeed, by the embedding $L^p \hookrightarrow M_{q,r}^p$ for $q < p < r$, $\text{supp } \mathcal{P}f \subset \overline{D_0}$, and triangle inequality, we have

$$\begin{aligned} \|\mathcal{P}f\|_{M_{q,r}^p} &\leq C \|\mathcal{P}f\|_{L^p(D_0)} \leq C \sum_{k \in K_0} \left\| \frac{1}{|\tau_k^{j_1}|} \int_{\tau_k^{j_1}} f dy \right\|_{L^p(\tau_k^{j_1})} \\ &\leq C \sum_{k \in K_0} |\tau_k^{j_1}|^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^q(\tau_k^{j_1})} \leq C 2^{d(j_1-j_0)(1-\frac{1}{r})} \|f\|_{M_{q,r}^p}. \end{aligned}$$

By the assumption (2.7),

$$(2.8) \quad \|f - \mathcal{P}f\|_{M_{q,r}^p} \leq \|f \mathbf{1}_{D_0^c}\|_{M_{q,r}^p} + \|f \mathbf{1}_{D_0} - \mathcal{P}f\|_{M_{q,r}^p} \leq \varepsilon + \|f \mathbf{1}_{D_0} - \mathcal{P}f\|_{M_{q,r}^p}$$

for any $f \in K$. We estimate $\|f\mathbf{1}_{D_0} - \mathcal{P}f\|_{M_{q,r}^p}$. Remark that

$$\|f\mathbf{1}_{D_0} - \mathcal{P}f\|_{M_{q,r}^p}^r = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} |\tau_k^j|^{\left(\frac{1}{p} - \frac{1}{q}\right)r} \|f\mathbf{1}_{D_0} - \mathcal{P}f\|_{L^q(\tau_k^j)}^r.$$

We first consider the case $j \leq j_0$. In this case, $\tau_k^j \cap D_0 \neq \emptyset$ if and only if $k \in \{-1, 0\}^d \subset \mathbb{Z}^d$. Further, for $k \in \{-1, 0\}^d$, $\tau_k^j \cap D_0 = \tau_k^{j_0}$. Therefore,

$$\begin{aligned} & \sum_{j \leq j_0} \sum_{k \in \mathbb{Z}^d} |\tau_k^j|^{\left(\frac{1}{p} - \frac{1}{q}\right)r} \|f\mathbf{1}_{D_0} - \mathcal{P}f\|_{L^q(\tau_k^j)}^r \\ &= \sum_{j \leq j_0} \sum_{k \in \{-1, 0\}^d} |\tau_k^j|^{\left(\frac{1}{p} - \frac{1}{q}\right)r} \|f - \mathcal{P}f\|_{L^q(\tau_k^{j_0})}^r \\ (2.9) \quad &= \sum_{j \leq j_0} 2^{-d(j-j_0)\left(\frac{1}{p} - \frac{1}{q}\right)r} \sum_{k \in \{-1, 0\}^d} |\tau_k^{j_0}|^{\left(\frac{1}{p} - \frac{1}{q}\right)r} \|f - \mathcal{P}f\|_{L^q(\tau_k^{j_0})}^r \\ &\leq C_{p,q,r,d} \sum_{k \in \{-1, 0\}^d} |\tau_k^{j_0}|^{\left(\frac{1}{p} - \frac{1}{q}\right)r} \|f - \mathcal{P}f\|_{L^q(\tau_k^{j_0})}^r \end{aligned}$$

since $q < p$. We next consider the case $j_0 \leq j \leq j_1$. If $j \geq j_0$ then, for each $\tau_k^j \in \mathcal{D}_j$, either $\tau_k^j \subset \tau_l^{j_0} \subset D_0$ for some $l \in \{-1, 0\}^d$ or $\tau_k^j \cap D_0 = \emptyset$ holds. Further, $\tau_k^j \subset \tau_l^{j_0}$ if and only if $k \in [-2^{j-j_0}, 2^{j-j_0}]^d$. We have

$$\begin{aligned} & \sum_{j_0 \leq j \leq j_1} \sum_{k \in \mathbb{Z}^d} |\tau_k^j|^{\left(\frac{1}{p} - \frac{1}{q}\right)r} \|f\mathbf{1}_{D_0} - \mathcal{P}f\|_{L^q(\tau_k^j)}^r \\ &= \sum_{j_0 \leq j \leq j_1} \sum_{k \in [-2^{j-j_0}, 2^{j-j_0}]^d} |\tau_k^j|^{\left(\frac{1}{p} - \frac{1}{q}\right)r} \|f - \mathcal{P}f\|_{L^q(\tau_k^j)}^r. \end{aligned}$$

For each $\tau_k^j \in \mathcal{D}_j$ in the summand of the right hand side, Hölder's inequality yields

$$\begin{aligned} \|f - \mathcal{P}f\|_{L^q(\tau_k^j)}^q &= \sum_{m \in \mathbb{Z}^d, \tau_m^{j_1} \subset \tau_k^j} \int_{\tau_m^{j_1}} \left| \frac{1}{|\tau_m^{j_1}|} \int_{\tau_m^{j_1}} (f(x) - f(y)) dy \right|^q dx \\ &\leq \sum_{m \in \mathbb{Z}^d, \tau_m^{j_1} \subset \tau_k^j} \frac{1}{|\tau_m^{j_1}|} \int_{\tau_m^{j_1}} \int_{\tau_m^{j_1}} |f(x) - f(y)|^q dy dx. \end{aligned}$$

We introduce the change of variable $y = x - z$. Since it holds for any $m \in \mathbb{Z}^d$ that $|x - y| \leq C_d 2^{-j_1}$ as long as $x, y \in \tau_m^{j_1}$, one has

$$\begin{aligned} \|f - \mathcal{P}f\|_{L^q(\tau_k^j)}^q &\leq \sum_{m \in \mathbb{Z}^d, \tau_m^{j_1} \subset \tau_k^j} \frac{1}{|\tau_m^{j_1}|} \int_{|z| \leq C 2^{-j_1}} \left(\int_{\tau_m^{j_1}} |f(x) - T(z)f(x)|^q dx \right) dz \\ &= 2^{j_1 d} \int_{|z| \leq C 2^{-j_1}} \|(T(z) - 1)f\|_{L^q(\tau_k^j)}^q dz \\ &\leq C 2^{j_1 d \frac{q}{r}} \left(\int_{|z| \leq C 2^{-j_1}} \|(T(z) - 1)f\|_{L^q(\tau_k^j)}^r dz \right)^{\frac{q}{r}}, \end{aligned}$$

where we have used Hölder's inequality in z to obtain the last line. Hence, combining above estimates, we reach to

$$\begin{aligned}
 (2.10) \quad & \sum_{j_0 \leq j \leq j_1} \sum_{k \in \mathbb{Z}^d} |\tau_k^j|^{\left(\frac{1}{p}-\frac{1}{q}\right)r} \|f \mathbf{1}_{D_0} - \mathcal{P}f\|_{L^q(\tau_k^j)}^r \\
 & \leq C_{q,r,d} \sum_{j_0 \leq j \leq j_1} \sum_{k \in \mathbb{Z}^d} |\tau_k^j|^{\left(\frac{1}{p}-\frac{1}{q}\right)r} 2^{j_1 d} \int_{|z| \leq C 2^{-j_1}} \|(T(z) - 1)f\|_{L^q(\tau_k^j)}^r dz \\
 & = C_{q,r,d} 2^{j_1 d} \int_{|z| \leq C 2^{-j_1}} \left(\sum_{j_0 \leq j \leq j_1} \sum_{k \in \mathbb{Z}^d} |\tau_k^j|^{\left(\frac{1}{p}-\frac{1}{q}\right)r} \|(T(z) - 1)f\|_{L^q(\tau_k^j)}^r \right) dz.
 \end{aligned}$$

We finally consider $j \geq j_1$. As in the previous case,

$$\begin{aligned}
 & \sum_{j \geq j_1} \sum_{k \in \mathbb{Z}^d} |\tau_k^j|^{\left(\frac{1}{p}-\frac{1}{q}\right)r} \|f \mathbf{1}_{D_0} - \mathcal{P}f\|_{L^q(\tau_k^j)}^r \\
 & = \sum_{j \geq j_1} \sum_{k \in [-2^j - j_0, 2^j - j_0]^d} |\tau_k^j|^{\left(\frac{1}{p}-\frac{1}{q}\right)r} \|f - \mathcal{P}f\|_{L^q(\tau_k^j)}^r.
 \end{aligned}$$

For each $\tau_k^j \in \mathcal{D}_j$ in the summand of the right hand side, $\tau_k^j \subset \tau_l^{j_1}$ for some $l \in K_0$. Denoting $l = l(k) \in K_0$ be a (unique) vector such that $\tau_k^j \subset \tau_l^{j_1}$, we have

$$\|f - \mathcal{P}f\|_{L^q(\tau_k^j)}^q \leq \frac{1}{|\tau_l^{j_1}|} \int_{\tau_k^j} \left(\int_{\tau_l^{j_1}} |f(x) - f(y)|^q dy \right) dx$$

as in the previous case. Now, we again introduce change of variable $y = x - z$. In this case, since $x \in \tau_k^j$ and $y \in \tau_l^{j_1}$, we have $|z| \leq C_d(2^{-j} + 2^{-j_1}) \leq C_d 2^{-j_1}$. The rest of the estimate is similar to the previous case. We obtain

$$\begin{aligned}
 (2.11) \quad & \sum_{j \geq j_1} \sum_{k \in \mathbb{Z}^d} |\tau_k^j|^{\left(\frac{1}{p}-\frac{1}{q}\right)r} \|f \mathbf{1}_{D_0} - \mathcal{P}f\|_{L^q(\tau_k^j)}^r \\
 & \leq C_{q,r,d} 2^{j_1 d} \int_{|z| \leq C 2^{-j_1}} \left(\sum_{j \geq j_1} \sum_{k \in \mathbb{Z}^d} |\tau_k^j|^{\left(\frac{1}{p}-\frac{1}{q}\right)r} \|(T(z) - 1)f\|_{L^q(\tau_k^j)}^r \right) dz.
 \end{aligned}$$

By (2.9), (2.10), and (2.11), we have

$$\begin{aligned}
 \|f \mathbf{1}_{D_0} - \mathcal{P}f\|_{M_{q,r}^p}^r & \leq C_{p,q,r,d} 2^{j_1 d} \int_{|z| \leq C_d 2^{-j_1}} \|(T(z) - 1)f\|_{M_{q,r}^p}^r dz \\
 & \leq C_{p,q,r,d} \left(\sup_{|z| \leq C_d 2^{-j_1}} \|(T(z) - 1)f\|_{M_{q,r}^p} \right)^r.
 \end{aligned}$$

Now, we choose j_1 so that $C_d 2^{-j_1} \leq 1/C(\varepsilon(C_{p,q,r,d})^{-1/r})$, where $C(\cdot)$ is the function in the assumption (2.7). Then, plugging the above estimate to (2.8) and using the assumption (2.7), we conclude that $\|f - \mathcal{P}f\|_{M_{q,r}^p} \leq 2\varepsilon$ for any $f \in K$ and so that $\|f - g\|_{M_{q,r}^p} \leq 4\varepsilon + \|\mathcal{P}f - \mathcal{P}g\|_{M_{q,r}^p}$ for any $f, g \in K$. This shows that if $\|\mathcal{P}f - \mathcal{P}g\|_{M_{q,r}^p} \leq \varepsilon$ then $\|f - g\|_{M_{q,r}^p} \leq 5\varepsilon$. Since \mathcal{P} is bounded

and since images of \mathcal{P} is finite dimensional, $\mathcal{P}K$ is totally bounded. Thus, we conclude from Lemma 2.21 that K is totally bounded.

Conversely, assume that a bounded set K is totally bounded and prove (2.7). By a standard argument, it suffices to show that each $f \in K \subset M_{q,r}^p$ satisfies

$$\|f \mathbf{1}_{\{|x| \geq R\}}\|_{M_{q,r}^p} + \sup_{|z| \leq 1/R} \|(T(z) - 1)f\|_{M_{q,r}^p} \rightarrow 0$$

as $R \rightarrow \infty$. Fix $f \in K$. Since $r < \infty$, for any $\varepsilon > 0$, there exists a finite set $\Omega \in \mathbb{Z} \times \mathbb{Z}^d$ such that

$$\sup_{|z| \leq 1} \left(\sum_{(j,k) \in \mathbb{Z}^{1+d} \setminus \Omega} |\tau_k^j|^{(\frac{1}{p} - \frac{1}{q})r} \|T(z)f\|_{L^q(\tau_k^j)}^r \right)^{1/r} \leq \varepsilon.$$

Hence, the proof is reduced to showing that

$$\|f \mathbf{1}_{\{|x| \geq R\}}\|_{L^q(\tau)} + \sup_{|z| \leq 1/R} \|(T(z) - 1)f\|_{L^q(\tau)} \rightarrow 0$$

as $R \rightarrow \infty$ for each dyadic cube τ and $f \in L_{\text{loc}}^q$. This is obvious. \square

3. COMPACTNESS TOOL

In this section, we treat a compactness result, a linear profile decomposition. We first collect notations and elementary facts in Section 3.1 and Section 3.2. The main result of this section is Theorem 3.11 in Section 3.3. Throughout this section, we suppose $d \geq 1$ and (1.4).

3.1. Deformations. We introduce a dilation

$$(D(h)f)(x) = h^{\frac{1}{\alpha}} f(hx), \quad h \in 2^{\mathbb{Z}},$$

translation in physical space

$$(T(a)f)(x) = f(x - a), \quad a \in \mathbb{R}^d,$$

translation in Fourier space

$$(P(b)f)(x) = e^{-ix \cdot b} f(x), \quad b \in \mathbb{R}^d,$$

and Schrödinger group $U(s) = e^{is\Delta}$, $s \in \mathbb{R}$. Each of them forms a group and inverses of them are summarized as follows:

$$D(h)^{-1} = D(h^{-1}), \quad T(a)^{-1} = T(-a), \quad P(b)^{-1} = P(-b), \quad U(s)^{-1} = U(-s).$$

It is easy to see that $T(a)$ and $U(s)$ are isometric bijection on $\hat{M}_{2,r}^{d\alpha}$ since they are just multiplication by $e^{-ia \cdot \xi}$ and $e^{-is|\xi|^2}$ respectively in the Fourier side. Similarly, $D(h)$ is also an isometric bijection on $\hat{M}_{2,r}^{d\alpha}$ as long as h is a dyadic number. For $P(b)$, we have the following.

Lemma 3.1. *It holds that*

$$2^{-d} \|f\|_{\hat{M}_{2,r}^{d\alpha}} \leq \|P(b)f\|_{\hat{M}_{2,r}^{d\alpha}} \leq 2^d \|f\|_{\hat{M}_{2,r}^{d\alpha}}$$

for any $b \in \mathbb{R}^d$ and $f \in \hat{M}_{2,r}^{d\alpha}$.

For the proof, see [32, Lemma 2.3].

Remark 3.2. Let $\ell_r : \hat{M}_{2,r}^{d\alpha} \rightarrow \mathbb{R}$ be as in (1.5). As an immediate consequence of Lemma 3.1, we see that $\ell_r(\cdot)$ is a quasi norm of $\hat{M}_{2,r}^{d\alpha}$ which is equivalent to $\|\cdot\|_{\hat{M}_{2,r}^{d\alpha}}$.

Next we collect commutation of the above operators:

$$\begin{aligned} D(h)T(a) &= T(h^{-1}a)D(h), & D(h)P(b) &= P(hb)D(h), \\ D(h)U(t) &= U(h^{-2}t)D(h), & T(a)P(b) &= e^{ia \cdot b}P(b)T(a), \\ T(a)U(s) &= U(s)T(a), \end{aligned}$$

and

$$U(s)P(b) = e^{-is|b|^2}P(b)U(s)T(-2sb).$$

The last one is nothing but a Galilean transform.

Definition 3.3. We call a bounded operator on $\hat{M}_{2,r}^{d\alpha}$ of the form

$$(3.1) \quad \mathcal{G} = e^{i\theta}D(h)P(b)U(s)T(a), \quad (\theta, h, s, a, b) \in \mathbb{R} \times 2^{\mathbb{Z}} \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$$

as a deformation of $\hat{M}_{2,r}^{d\alpha}$. We define $G \subset \mathcal{L}(\hat{M}_{2,r}^{d\alpha})$ as a set of all deformations. G forms a group with functional composition as a multiplication. $\text{Id} \in G$ is the identity element.

It follows from Lemma 3.1 and the above commutation that

$$2^{-d} \|f\|_{\hat{M}_{2,r}^{d\alpha}} \leq \|\mathcal{G}f\|_{\hat{M}_{2,r}^{d\alpha}} \leq 2^d \|f\|_{\hat{M}_{2,r}^{d\alpha}}$$

and

$$\ell_r(\mathcal{G}f) = \ell_r(f)$$

for any $\mathcal{G} \in G$ and $f \in \hat{M}_{2,r}^{d\alpha}$.

Remark 3.4 (Normal representation). By the commutation relations above, we can freely change the order of four operators D , P , U , and T in the representation (3.1) by a suitable change of parameters. We refer the representation as in (3.1) to as a *normal representation* of $\mathcal{G} \in G$.

3.2. Orthogonality of families of deformations. We next introduce several notions on families of deformations.

Definition 3.5 (a vanishing family). We say a family of deformations $\{\mathcal{G}_n\}_n \subset G$ is vanishing if, in the normal representation

$$\mathcal{G}_n = e^{i\theta_n}D(h_n)P(b_n)U(s_n)T(a_n),$$

it holds that

$$|\log h_n| + |b_n| + |s_n| + |a_n| \rightarrow \infty$$

as $n \rightarrow \infty$.

Lemma 3.6. A family $\{\mathcal{G}_n\}_n \subset G$ is vanishing if and only if $\{\mathcal{G}_n^{-1}\}_n$ is vanishing.

Proof. If we denote $\mathcal{G}_n = e^{i\theta_n}D(h_n)P(b_n)U(s_n)T(a_n)$ then

$$\mathcal{G}_n^{-1} = e^{i(-\theta_n + a_n \cdot b_n + s_n |b_n|^2)} D(h_n^{-1})P(-h_n b_n)U\left(-\frac{s_n}{h_n^2}\right)T\left(-\frac{a_n + 2s_n b_n}{h_n}\right).$$

It is obvious that if \mathcal{G}_n is not vanishing then \mathcal{G}_n^{-1} is not vanishing. The other direction follows from the same argument by the relation $(\mathcal{G}_n^{-1})^{-1} = \mathcal{G}_n$. \square

Lemma 3.7. *If a family $\{\mathcal{G}_n\}_n \subset G$ is not vanishing then there exist a subsequence n_k of n and $\mathcal{G} \in G$ such that $\mathcal{G}_{n_k} \rightarrow \mathcal{G}$ strongly in $\mathcal{L}(\hat{M}_{2,r}^{d\alpha})$ as $k \rightarrow \infty$, i.e., for any $\phi \in \hat{M}_{2,r}^{d\alpha}$, $\mathcal{G}_{n_k}\phi \rightarrow \mathcal{G}\phi$ (strongly) in $\hat{M}_{2,r}^{d\alpha}$ as $k \rightarrow \infty$.*

Proof. We denote $\mathcal{G}_n = e^{i\theta_n} D(h_n)P(b_n)U(s_n)T(a_n)$. Since \mathcal{G}_n is not vanishing, there exists a subsequence n_k such that $(e^{i\theta_{n_k}}, h_{n_k}, b_{n_k}, s_{n_k}, a_{n_k})$ converges to $(e^{i\theta}, h, b, s, a) \in \{z \in \mathbb{C} \mid |z| = 1\} \times 2^{\mathbb{Z}} \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ as $k \rightarrow \infty$. The conclusion is obvious by taking $\mathcal{G} := e^{i\theta} D(h)P(b)U(s)T(a)$. \square

Proposition 3.8. *For a family $\{\mathcal{G}_n\}_n \subset G$ of deformations, the following three statements are equivalent:*

- (1) $\{\mathcal{G}_n\}_n$ is vanishing;
- (2) For any $\phi \in \hat{M}_{2,r}^{d\alpha}$, $\mathcal{G}_n\phi \rightarrow 0$ weakly-* in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$;
- (3) For any subsequence n_k of n there exist a sequence $\{u_k\}_k \subset \hat{M}_{2,r}^{d\alpha}$ and subsequence k_l of k such that $u_{k_l} \rightarrow 0$ and $\mathcal{G}_{n_{k_l}}^{-1}u_{k_l} \rightarrow \phi \neq 0$ weakly-* in $\hat{M}_{2,r}^{d\alpha}$ as $l \rightarrow \infty$.

Proof. “(2) \Rightarrow (3)” is obvious by taking $u_k = \mathcal{G}_{n_k}\phi$ for some $\phi \neq 0$. “(3) \Rightarrow (1)” is also immediate because the contraposition is Lemma 3.7.

Let us prove “(1) \Rightarrow (2)”. By density argument, it suffices to show that $(\mathcal{F}\mathcal{G}_n\phi, \mathcal{F}\psi) \rightarrow 0$ as $n \rightarrow \infty$ for any $\phi, \psi \in \mathcal{F}(C_0^\infty) \subset \mathcal{S}$. If $|\log h_n| \rightarrow \infty$ then we use Hölder’s inequality to obtain $|\mathcal{F}\mathcal{G}_n\phi, \mathcal{F}\psi| \leq \|\mathcal{F}\mathcal{G}_n\phi\|_{L^{r'}} \|\mathcal{F}\psi\|_{L^r} = C_{\phi,\psi}(h_n)^{\frac{1}{\alpha} - \frac{d}{r}}$. We obtain the result by taking $r > d\alpha$ if $h_n \rightarrow 0$ and $r < d\alpha$ if $h_n \rightarrow \infty$. Let us next suppose that $|\log h_n|$ is bounded and $|b_n| \rightarrow \infty$ as $n \rightarrow \infty$. In this case, we have $(\mathcal{F}\mathcal{G}_n\phi, \mathcal{F}\psi) = 0$ for large n because $\mathcal{F}\phi$ and $\mathcal{F}\psi$ have compact support and because $|\log h_n|$ is bounded. Let us suppose that $|\log h_n| + |b_n|$ is bounded and $|s_n| \rightarrow \infty$ as $n \rightarrow \infty$. In this case, the result follows from

$$|(\mathcal{F}\mathcal{G}_n\phi, \mathcal{F}\psi)| = |(\mathcal{G}_n\phi, \psi)| \leq C \|U(s_n)\phi\|_{L^\infty} \|\psi\|_{L^1} \leq C |s_n|^{-\frac{d}{2}} \|\phi\|_{L^1} \|\psi\|_{L^1},$$

where the constant C depends on the bound of $|\log h_n|$. Finally, let us consider the case where $|\log h_n| + |b_n| + |s_n|$ is bounded and $|a_n| \rightarrow \infty$ as $n \rightarrow \infty$. Thanks to the boundedness of $|\log h_n| + |b_n| + |s_n|$, the proof boils down to showing that $(T(a_n)\phi, \psi) \rightarrow 0$ as $n \rightarrow \infty$, which is obvious. \square

Let us now introduce a notion of orthogonality of two families of deformations.

Definition 3.9 (Orthogonality). *Let $\{\mathcal{G}_n\}_n, \{\tilde{\mathcal{G}}_n\}_n \subset G$ be two families of deformations. We say $\{\mathcal{G}_n\}_n$ and $\{\tilde{\mathcal{G}}_n\}_n$ are orthogonal if $\{\mathcal{G}_n^{-1}\tilde{\mathcal{G}}_n\}_n$ is vanishing.*

Proposition 3.10. *We introduce the following relation \sim for families of deformations: For $\{\mathcal{G}_n\}_n, \{\tilde{\mathcal{G}}_n\}_n \subset G$, $\{\mathcal{G}_n\}_n \sim \{\tilde{\mathcal{G}}_n\}_n$ is true if $\{\mathcal{G}_n\}_n$ and $\{\tilde{\mathcal{G}}_n\}_n$ are not orthogonal. Then, \sim defines an equivalent relation.*

Proof. The reflexivity of \sim follows from the fact that sequence of the identity $\{\mathcal{G}_n = \text{Id}\}_n$ is not vanishing. The symmetry of \sim follows from Lemma 3.6. The transitivity of \sim is a consequence of Lemma 3.7. Indeed, if $\{\mathcal{G}_n^1\}_n \sim \{\mathcal{G}_n^2\}_n$ and $\{\mathcal{G}_n^2\}_n \sim \{\mathcal{G}_n^3\}_n$ then there exists a subsequence n_k such that

$$(\mathcal{G}_{n_k}^1)^{-1}\mathcal{G}_{n_k}^2 \rightarrow \mathcal{G} \in G, \quad (\mathcal{G}_{n_k}^2)^{-1}\mathcal{G}_{n_k}^3 \rightarrow \tilde{\mathcal{G}} \in G$$

strongly in $\mathcal{L}(\hat{M}_{2,r}^{d\alpha})$ as $k \rightarrow \infty$, in light of Lemma 3.7. For the same subsequence n_k , we have

$$(\mathcal{G}_{n_k}^1)^{-1} \mathcal{G}_{n_k}^3 = [(\mathcal{G}_{n_k}^1)^{-1} \mathcal{G}_{n_k}^2][(\mathcal{G}_{n_k}^2)^{-1} \mathcal{G}_{n_k}^3] \rightarrow \mathcal{G} \tilde{\mathcal{G}} \in G$$

strongly in $\mathcal{L}(\hat{M}_{2,r}^{d\alpha})$ as $k \rightarrow \infty$. This implies that the sequence $\{(\mathcal{G}_n^1)^{-1} \mathcal{G}_n^3\}_n$ does not satisfy the third assertion of Proposition 3.8. \square

We conclude this section with an explicit representation of orthogonality. Let $\{\mathcal{G}_n^j = e^{i\theta_n^j} D(h_n^j) P(b_n^j) U(s_n^j) T(a_n^j)\}_n \subset G$ ($j = 1, 2$) be families of deformations in the normal representation. $\{\mathcal{G}_n^1\}_n$ and $\{\mathcal{G}_n^2\}_n$ are orthogonal if and only if

$$(3.2) \quad \left| \log \frac{h_n^1}{h_n^2} \right| + \left| b_n^1 - \frac{h_n^2}{h_n^1} b_n^2 \right| + \left| s_n^1 - \left(\frac{h_n^1}{h_n^2} \right)^2 s_n^2 \right| + \left| a_n^1 - \frac{h_n^1}{h_n^2} a_n^2 + 2 \left(\frac{h_n^1}{h_n^2} \right)^2 s_n^2 \left(b_n^1 - \frac{h_n^2}{h_n^1} b_n^2 \right) \right| \rightarrow \infty$$

as $n \rightarrow \infty$. This is immediate from the identity

$$(\mathcal{G}_n^2)^{-1} \mathcal{G}_n^1 = e^{i\theta_n} D \left(\frac{h_n^1}{h_n^2} \right) P \left(b_n^1 - \frac{h_n^2}{h_n^1} b_n^2 \right) U \left(s_n^1 - \left(\frac{h_n^1}{h_n^2} \right)^2 s_n^2 \right) T \left(a_n^1 - \frac{h_n^1}{h_n^2} a_n^2 + 2 \left(\frac{h_n^1}{h_n^2} \right)^2 s_n^2 \left(b_n^1 - \frac{h_n^2}{h_n^1} b_n^2 \right) \right)$$

with suitable $\theta_n \in \mathbb{R}$.

3.3. Linear profile decomposition.

Theorem 3.11 (Linear profile decomposition). *Let $\frac{2}{d+\frac{2}{d+3}} < \alpha < 2/d$ and $(d\alpha)' < r < ((d+2)\alpha)^*$. For any bounded sequence $\{u_n\}_n \subset \hat{M}_{2,r}^{d\alpha}$, there exist $\phi^j \in \hat{M}_{2,r}^{d\alpha}$, $R_n^j \in \hat{M}_{2,r}^{d\alpha}$ and pairwise orthogonal families of deformations $\{\mathcal{G}_n^j\}_n \subset G$ ($j = 1, 2, \dots$) parametrized as in (3.1) by $\{\Gamma_n^j = (0, h_n^j, \xi_n^j, s_n^j, y_n^j)\}_n$ such that, extracting a subsequence in n ,*

$$(3.3) \quad u_n = \sum_{j=1}^l \mathcal{G}_n^j \phi^j + R_n^l$$

for all $n, l \geq 1$. Moreover, $\{R_n^j\}_{n,j}$ satisfies

$$(3.4) \quad (\mathcal{G}_n^k)^{-1} R_n^j \rightharpoonup \begin{cases} \phi^k & j < k, \\ 0 & j \geq k \end{cases}$$

weakly-* in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$ for all $j \geq 0$ and $k \geq 1$, with a convention $R_n^0 = u_n$, and

$$(3.5) \quad \limsup_{n \rightarrow \infty} \left\| e^{it\Delta} R_n^l \right\|_{L_{t,x}^{(d+2)\alpha}(\mathbb{R}^{1+d})} \rightarrow 0$$

as $l \rightarrow \infty$. Furthermore, a decoupling inequality

$$(3.6) \quad \limsup_{n \rightarrow \infty} \ell_r(u_n) \geq \left(\sum_{j=1}^J \ell_r(\phi^j)^r \right)^{1/r} + \limsup_{n \rightarrow \infty} \ell_r(R_n^J)$$

holds for all $J \geq 1$.

The proof is done by modifying the argument in the L^2 case [3, 6]. The modification to the $\hat{M}_{2,r}^{d\alpha}$ -framework is essentially the same as for the Airy equation, see [32]. For self-containedness, we give a proof in Appendix B.

4. PROOF OF THEOREM 1.15

We first introduce a function $L(E)$ for $E \geq 0$ by

$$L(E) = \sup \left\{ S_I(u) \left| \begin{array}{l} u(t) : I \times \mathbb{R}^d \rightarrow \mathbb{C} : \text{sol. to (1.1),} \\ \sup_{t \in I} \ell_r(u(t)) \leq E \end{array} \right. \right\} \in [0, \infty].$$

Remark that, in the above definition, $u(t)$ is not always a maximal-lifespan solution. Small data scattering implies that $L(E) \leq CE$ for $E \leq \delta$. Further, since $Q_\alpha(t, x)$ is a nonscattering solution, $L(\ell_r(Q)) = \infty$. By the long time stability, we see that $L(E)$ is continuous. Combining these facts, one sees that there exists a critical value

$$\begin{aligned} E_c &= E_c(\alpha, \hat{M}_{2,r}^{d\alpha}, \ell_r) := \sup\{E \mid L(E) < \infty\} \\ &= \min\{E \mid L(E) = \infty\} \in [\delta, \ell_r(Q)]. \end{aligned}$$

By definition, one has

$$(4.1) \quad E_c \leq E_2.$$

Indeed, by definition of E_2 , for any $\varepsilon > 0$, there exists a solution $v(t)$ with maximal interval I that does not scatter for positive time direction and

$$\limsup_{t \uparrow \sup I} \ell_r(v(t)) \leq E_2 + \varepsilon.$$

Then, one can choose $t_0 \in I$ so that

$$\sup_{t \in [t_0, \sup I)} \ell_r(v(t)) \leq E_2 + 2\varepsilon.$$

On the other hand, since $v(t)$ does not scatter for positive time direction, $L(\sup_{t \in [t_0, \sup I)} \ell_r(v(t))) = \infty$, proving

$$E_c \leq \sup_{t \in [t_0, \sup I)} \ell_r(v(t))$$

and so $E_c \leq E_2 + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we obtain (4.1).

Our task is now to show

Theorem 4.1. *Let $d \geq 1$ and suppose (1.4). There exists a solution $v(t)$ to (1.1) with maximal existence interval such that*

- (1) $v(t)$ does not scatter for both time directions.
- (2) $\sup_{I_{\max}(v) \cap \{t \geq 0\}} \ell_r(v(t)) = \sup_{I_{\max}(v) \cap \{t \leq 0\}} \ell_r(v(t)) = E_c(\alpha, \hat{M}_{2,r}^{d\alpha}, \ell_r).$
- (3) $v(t)$ is almost periodic modulo symmetry as in (1.6).

As an immediate consequence of this theorem, we obtain $E_2 = E_c$. Indeed, once we obtain a solution $v(t)$ with the first two properties of the above theorem, it follows that

$$E_2 \leq \limsup_{t \uparrow \sup I_{\max}(v)} \ell_r(v(t)) \leq \sup_{I_{\max}(v) \cap \{t \geq 0\}} \ell_r(v(t)) = E_c.$$

By means of (4.1), we obtain the desired result.

4.1. The key convergence result. For $u : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ and $\tau \in I$, we denote $S_{\geq \tau}(u) := S_{I \cap \{t \geq \tau\}}(u)$, $S_{\leq \tau}(u) := S_{I \cap \{t \leq \tau\}}(u)$.

Proposition 4.2. *Let $u_n : I_n \times \mathbb{R}^d \rightarrow \mathbb{C}$ be a sequence of solutions to (1.1) such that*

$$(4.2) \quad \limsup_{n \rightarrow \infty} \sup_{t \in I_n} \ell_r(u_n(t)) = E_c,$$

and let $t_n \in I_n$ be a sequence of times such that

$$(4.3) \quad \lim_{n \rightarrow \infty} S_{\geq t_n}(u_n) = \lim_{n \rightarrow \infty} S_{\leq t_n}(u_n) = \infty.$$

Then, there exist a sequence of deformations $\mathcal{G}_n = \{D(h_n)P(b_n)T(a_n)\}_n$ and a subsequence of n such that $(\mathcal{G}_n)^{-1}u_n(t_n)$ converges strongly in $\hat{M}_{2,r}^{d\alpha}$ to a function $\phi \in \hat{M}_{2,r}^{d\alpha}$ along the subsequence. Further, a solution $\Phi(t)$ of (1.1) with $\Phi(0) = \phi$ satisfies the first two properties in Theorem 4.1.

In the rest of this section, we prove this proposition. Our argument is in the same spirit as in [26]. By the time translation symmetry of (1.1), we may let $t_n = 0$. We apply profile decomposition lemma to yield a decomposition

$$u_n(0) = \sum_{j=1}^J \mathcal{G}_n^j \phi^j + w_n^J$$

up to subsequence, where \mathcal{G}_n^j is parameterized as in (3.1) with $\theta_n^j \equiv 0$.

Refining the subsequence and changing notations, we may assume that for each j , the sequence $\{s_n^j\}$ converges to some $s^j \in \{0, \pm\infty\}$. Further, if $s^j = 0$ then we may let $s_n^j \equiv 0$. Let $\Phi^j : I^j \times \mathbb{R}^N \rightarrow \mathbb{C}$ be a nonlinear profile associated with $(\phi^j, \{s_n^j\}_n)$, i.e.,

- If $s^j = 0$ then $\Phi^j(t)$ is a solution to (1.1) with $\Phi^j(0) = \phi^j$.
- If $s^j = \infty$ (resp. $s^j = -\infty$) then $\Phi^j(t)$ is a solution to (1.1) that scatters to ϕ^j for positive time direction (resp. negative time direction).

Define

$$v_n^j(t) := e^{-i(h_n^j)^2 |b_n^j|^2 t} D(h_n^j) P(b_n^j) T(a_n^j - 2b_n^j (h_n^j)^2 t) \Phi^j((h_n^j)^2 t + s_n^j)$$

and

$$\tilde{u}_n^J(t) := \sum_{j=1}^J v_n^j(t) + U(t)w_n^J.$$

Remark that v_n^j solves (1.1) and $v_n^j(0) = D(h_n^j)P(b_n^j)T(a_n^j)\Phi^j(s_n^j)$.

Lemma 4.3. *There exists j such that $\Phi^j(t)$ does not scatter for positive time direction.*

Proof of Lemma 4.3. Assume for contradiction that Φ^j scatters for positive time direction for all j . We apply long time stability with $\tilde{u}(t) = \tilde{u}_n^J(t)$ for large J and $n \geq N(J)$.

We first demonstrate that $\tilde{u}_n^J(0) - u_n(0) \rightarrow 0$ in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$. Indeed, we have

$$\begin{aligned} \tilde{u}_n^J(0) - u_n(0) &= \sum_{j=1}^J v_n^j(0) - \mathcal{G}_n^j \phi^j \\ &= \sum_{j=1}^J D(h_n^j) P(b_n^j) T(a_n^j) U(s_n^j) (U(-s_n^j) \Phi^j(s_n^j) - \phi^j). \end{aligned}$$

Hence, by definition of nonlinear profile $\Phi^j(t)$ and Lemma 3.1,

$$\|\tilde{u}_n^J(0) - u_n(0)\|_{\hat{M}_{2,r}^{d\alpha}} \leq 2^d \sum_{j=1}^J \|U(-s_n^j) \Phi^j(s_n^j) - \phi^j\|_{\hat{M}_{2,r}^{d\alpha}} \rightarrow 0$$

as $n \rightarrow \infty$.

We will show that $\|\tilde{u}_n^J\|_{L_{t,x}^{(d+2)\alpha}(\mathbb{R}_+ \times \mathbb{R})}$ is uniformly bounded and that the error $e := i\partial_t \tilde{u}_n^J + \Delta \tilde{u}_n^J + |\tilde{u}_n^J|^{2\alpha} \tilde{u}_n^J$ tends to zero as $n \rightarrow \infty$ for each J . By the orthogonality, it follows that

$$\left\| |v_n^j|^\theta |v_n^k|^{1-\theta} \right\|_{L_{t,x}^{\frac{(d+2)\alpha}{2}}} \rightarrow 0$$

as $n \rightarrow \infty$ for any $j \neq k$ and $0 < \theta < 1$ (see [2, 35]). Hence, we see that

$$\|e\|_{L_{t,x}^{\frac{(d+2)\alpha}{2\alpha+1}}(\mathbb{R}_+ \times \mathbb{R})} \rightarrow 0$$

as $n \rightarrow \infty$. On the other hand, since $\sum_{j=1}^\infty \|\phi^j\|_{\hat{M}_{2,r}^{d\alpha}}^r < \infty$, for any $\varepsilon > 0$ there exists $J_0 = J_0(\varepsilon)$ such that

$$\left\| \sum_{j=J_0}^J \mathcal{G}_n^j \phi^j \right\|_{\hat{M}_{2,r}^{d\alpha}} \leq \varepsilon$$

for any $J > J_0$ and $n > N(J)$, where we have used Lemma 3.1 and orthogonality of \mathcal{G}_n^j . This implies that

$$\left\| \sum_{j=J_0}^J v_n^j(0) \right\|_{\hat{M}_{2,r}^{d\alpha}} \leq 2\varepsilon$$

for any $J > J_0(\varepsilon)$ and $n > N(J)$. By small data result, one then sees that

$$\|\tilde{v}_n^J\|_{L^\infty(\mathbb{R}, \hat{M}_{2,r}^{d\alpha}) \cap L_{t,x}^{(d+2)\alpha}(\mathbb{R} \times \mathbb{R}^d)} \leq C\varepsilon$$

for such J and n , provided $\varepsilon > 0$ is sufficiently small, where $\tilde{v}_n^J(t)$ is a solution to (1.1) with $\tilde{v}_n^J(0) = \sum_{j=J_0(\varepsilon)}^J v_n^j(0)$. As in the long time perturbation, it follows that

$$\left\| \sum_{j=J_0}^J v_n^j \right\|_{L^\infty(\mathbb{R}, \hat{M}_{2,r}^{d\alpha}) \cap L_{t,x}^{(d+2)\alpha}(\mathbb{R} \times \mathbb{R}^d)} \leq C'\varepsilon$$

for any $J > J_0$ and $n > N'(J)$. Hence,

$$\|u_n^J\|_{L_{t,x}^{(d+2)\alpha}} \leq \sum_{j=1}^{J_0} \|\Phi^j\|_{L_{t,x}^{(d+2)\alpha}} + \left\| \sum_{j=J_0}^J v_n^j \right\|_{L_{t,x}^{(d+2)\alpha}} + C \|w_n^J\|_{\hat{M}_{2,r}^{d\alpha}} < \infty$$

for any $J > J_0$ and $n \geq N''(J)$. This contradicts with the assumption (4.3). \square

By the previous result, there exists at least one Φ^j that blows up for positive time direction. Renumbering the index j if necessary, we may assume that Φ^j does not scatter for positive time direction if and only if $1 \leq j \leq J_1$. Remark that the number J_1 is finite because of decoupling inequality and small data scattering. Also remark that $s^j \neq \infty$ for $1 \leq j \leq J_1$ otherwise it scatters for positive time direction by definition of Φ^j .

We now prove that $J_1 = 1$. For each $m, n \geq 1$ let us define an integer $j(m, n) \in \{1, 2, \dots, J_1\}$ and an interval K_n^m of the form $[0, \tau]$ by

$$\sup_{1 \leq j \leq J_1} S_{K_n^m}(v_n^j) = S_{K_n^m}(v_n^{j(m,n)}) = m.$$

By the pigeonhole principle, there is a $j_1 \in \{1, 2, \dots, J_1\}$ so that for infinitely many m one has $j(m, n) = j_1$ for infinitely many n . By reordering the indices, we may assume that $j_1 = 1$. Then, by definition of E_2 and (4.1),

$$(4.4) \quad \limsup_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{t \in K_n^m} \ell_r(v_n^1(t)) \geq E_2 \geq E_c.$$

Lemma 4.4. $\psi^j \equiv 0$ for $j \geq 2$. And, $w_n^1 \rightarrow 0$ in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$.

Proof of Lemma 4.4. In light of long time stability, it holds for each m that

$$(4.5) \quad \limsup_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{t \in K_n^m} \|u_n(t) - \tilde{u}_n^J(t)\|_{\hat{M}_{2,r}^{d\alpha}} = 0.$$

Here we remark that, by definition of K_n^m , we have

$$\sup_n S_{K_n^m}(v_n^j) \leq m$$

for $1 \leq j \leq J_1$ and so the assumption of long time stability is fulfilled. For $j > J_1$, we have $S_{\geq 0}(v_n^j) \leq C_j < \infty$. Define

$$c_j := \inf_{t \in I_{\max}(\Phi^j)} \ell_r(\Phi^j(t)).$$

We shall show that $c_j = 0$ for $j \geq 2$.

Assume for contradiction that $c_{j_0} > 0$ for some $j_0 \geq 2$. By means of (4.4), for any $\varepsilon > 0$ there exists $m = m(\varepsilon)$ such that

$$\sup_{t \in K_n^m} \ell_r(v_n^1(t))^r \geq E_c^r - \varepsilon$$

for a subsequence of n . The subsequence depends on m and is denoted again by n .

Fix m . Then, for the same ε , we can choose $J = J(\varepsilon, m) = J(\varepsilon)$ so that

$$\sup_{t \in K_n^m} \|u_n(t) - \tilde{u}_n^J\|_{\hat{M}_{2,\sigma}^{d\alpha}} \leq \varepsilon$$

as long as $n \geq N(\varepsilon, m, J) = N(\varepsilon)$ by using (4.5). Without loss of generality, we may assume that $J > j_0$.

Fix J . Set

$$\tilde{w}_n^k(t) := \sum_{j=k+1}^J v_n^j(t) + e^{it\Delta} w_n^J.$$

for $k = 0, 1, 2, \dots, J-1$ and $\tilde{w}_n^J(t) = e^{it\Delta} w_n^J$. Remark that $\tilde{w}_n^0 = \tilde{u}_n^J$. For each $j = 0, 1, 2, \dots, J$, we may show that for any given sequence $\{t_n\}_n$ such that $t_n \in K_n^m$ there exists a subsequence of n , which is again denoted by n , such that

$$(4.6) \quad \ell_r(\tilde{w}_n^j(t_n))^r - \gamma \ell_r(v_n^{j+1}(t_n))^r - \gamma \ell_r(\tilde{w}_n^{j+1}(t_n))^r \geq o_n(1)$$

for any $0 < \gamma < 1$.

Before the proof of (4.6), we shall complete the proof of the lemma. Once inequality (4.6) is proven, we deduce for any $\{t_n\}_n$ with $t_n \in K_n^m$ that

$$(4.7) \quad \ell_r(\tilde{u}_n^J(t_n))^r - \sum_{j=1}^J \gamma^j \ell_r(v_n^j(t_n))^r - \gamma^J \ell_r(\tilde{w}_n^J(t_n))^r \geq o_n(1)$$

holds up to subsequence. Now, choose a sequence $\{t_n\}_n$ so that

$$\ell_r(v_n^1(t_n))^r \geq \sup_{t \in K_n^m} \ell_r(v_n^1)^r - \frac{\varepsilon}{2}.$$

Then, by means of (4.7), extracting subsequence of n , one verifies that

$$\sup_{t \in K_n^m} \ell_r(\tilde{u}_n^J)^r \geq \gamma \sup_{t \in K_n^m} \ell_r(v_n^1)^r + \sum_{j=2}^J \gamma^j c_j^r + \gamma^J \ell_r(w_n^J)^r - \frac{\gamma}{2} \varepsilon + o_n(1),$$

where we have used

$$\ell_r(\tilde{u}_n^J(t_n))^r \leq \sup_{t \in K_n^m} \ell_r(\tilde{u}_n^J(t))^r,$$

$\ell_r(v_n^j(t)) \geq c_j$ for $j \geq 2$, and $\ell_r(\tilde{w}_n^J(t_n)) = \ell_r(w_n^J)$. Hence, by definition of m , for large n ,

$$\sup_{t \in K_n^m} \ell_r(\tilde{u}_n^J)^r \geq \gamma \sup_{t \in K_n^m} \ell_r(v_n^1)^r + \sum_{j=2}^J \gamma^j c_j^r - \varepsilon \geq \gamma E_c^r + \gamma^{j_0} c_{j_0}^r - 2\varepsilon.$$

By assumption (4.2), we also have

$$\sup_{t \in K_n^m} \ell_r(u_n(t)) \leq E_c + \varepsilon$$

for large n . Thus, for sufficiently large n , we have

$$\begin{aligned} E_c + \varepsilon &\geq \sup_{t \in K_n^m} \ell_r(u_n(t)) \\ &\geq \sup_{t \in K_n^m} \ell_r(\tilde{u}_n^J(t)) - C \sup_{t \in K_n^m} \|u_n(t) - \tilde{u}_n^J(t)\|_{\dot{M}_{2,r}^{d\alpha}} \\ &\geq (\gamma E_c^r + \gamma^{j_0} c_{j_0}^r - 2\varepsilon)^{\frac{1}{r}} - C\varepsilon, \end{aligned}$$

that is,

$$c_{j_0}^r \leq C\varepsilon + \gamma^{-j_0} (1 - \gamma) E_c^r,$$

which is a contradiction when ε is sufficiently small and γ is sufficiently close to one. Hence, $\phi^j \equiv 0$ for $j \geq 2$. Once we know $\phi^j \equiv 0$ for $j \geq 2$, we see that $w_n^J = w_n^1$ for all J . Arguing as above, one sees that

$$\limsup_{n \rightarrow \infty} \|w_n^1\|_{\hat{M}_{2,r}^{d\alpha}} = \limsup_{J \rightarrow \infty} \limsup_{n \rightarrow \infty} \|w_n^J\|_{\hat{M}_{2,r}^{d\alpha}} \leq \varepsilon$$

for any $\varepsilon > 0$. Hence, $\lim_{n \rightarrow \infty} \|w_n^1\|_{\hat{M}_{2,r}^{d\alpha}} = 0$.

Thus, it suffices to show (4.6) to complete the proof. We first note that $\tilde{w}_n^j = v_n^j + \tilde{w}_n^{j+1}$ and so

$$\begin{aligned} \|\mathcal{F}\tilde{w}_n^j(t_n)\|_{L^2(\tau)}^2 &= \|\mathcal{F}v_n^{j+1}(t_n)\|_{L^2(\tau)}^2 + \|\mathcal{F}\tilde{w}_n^{j+1}(t_n)\|_{L^2(\tau)}^2 \\ &\quad + 2\operatorname{Re} \langle \mathcal{F}v_n^{j+1}(t_n), \mathcal{F}\tilde{w}_n^{j+1}(t_n) \rangle_\tau \end{aligned}$$

for each dyadic cube $\tau \in \mathcal{D}$, where $\langle f, g \rangle_\tau = \int_\tau f(x)\bar{g}(x)dx$. By an elementary inequality

$$(a - b)^{\frac{\sigma}{2}} \geq \left(\frac{m}{m+1} \right)^{\frac{\sigma-2}{2}} a^{\frac{\sigma}{2}} - m^{\frac{\sigma-2}{2}} b^{\frac{\sigma}{2}}$$

for any $a \geq b \geq 0$ and $m > 0$ and by embedding $\ell_{\mathcal{D}}^2 \hookrightarrow \ell_{\mathcal{D}}^r$, it follows that

$$\begin{aligned} &\sum_{\tau \in \mathcal{D}} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} \|\tilde{w}_n^j\|_{L^2(\tau)}^r \\ &= \sum_{\tau \in \mathcal{D}} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} \left(\|\mathcal{F}v_n^{j+1}\|_{L^2(\tau)}^2 + \|\mathcal{F}\tilde{w}_n^{j+1}\|_{L^2(\tau)}^2 + 2\operatorname{Re} \langle \mathcal{F}v_n^{j+1}, \mathcal{F}\tilde{w}_n^{j+1} \rangle_\tau \right)^{\frac{r}{2}} \\ &\geq \sum_{\tau \in \mathcal{D}} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} \left(\|\mathcal{F}v_n^{j+1}\|_{L^2(\tau)}^2 + \|\mathcal{F}\tilde{w}_n^{j+1}\|_{L^2(\tau)}^2 - 2|\langle \mathcal{F}v_n^{j+1}, \mathcal{F}\tilde{w}_n^{j+1} \rangle_\tau| \right)^{\frac{r}{2}} \\ &\geq \left(\frac{m}{m+1} \right)^{\frac{r-2}{2}} \sum_{\tau \in \mathcal{D}} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} \left(\|\mathcal{F}v_n^{j+1}\|_{L^2(\tau)}^2 + \|\mathcal{F}\tilde{w}_n^{j+1}\|_{L^2(\tau)}^2 \right)^{\frac{r}{2}} \\ &\quad - 2^{\frac{r}{2}} m^{\frac{r-2}{2}} \sum_{\tau \in \mathcal{D}} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} |\langle \mathcal{F}v_n^{j+1}, \mathcal{F}\tilde{w}_n^{j+1} \rangle_\tau|^{\frac{r}{2}} \\ &\geq \left(\frac{m}{m+1} \right)^{\frac{r-2}{2}} \left(\sum_{\tau \in \mathcal{D}} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} \|\mathcal{F}v_n^{j+1}\|_{L^2(\tau)}^r + \sum_{\tau \in \mathcal{D}} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} \|\mathcal{F}\tilde{w}_n^{j+1}\|_{L^2(\tau)}^r \right) \\ &\quad - 2^{\frac{r}{2}} m^{\frac{r-2}{2}} \sum_{\tau \in \mathcal{D}} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} |\langle \mathcal{F}v_n^{j+1}, \mathcal{F}\tilde{w}_n^{j+1} \rangle_\tau|^{\frac{r}{2}}, \end{aligned}$$

where we have omitted the time variable (t_n) in the above estimate. Hence, the equation (4.7) follows if we show

$$(4.8) \quad \sum_{\tau \in \mathcal{D}} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} |\langle \mathcal{F}v_n^{j+1}(t_n), \mathcal{F}\tilde{w}_n^{j+1}(t_n) \rangle_\tau|^{\frac{r}{2}} \rightarrow 0$$

as $n \rightarrow \infty$ up to subsequence.

We now claim that it suffices to show the above convergence with replacing $v_n^{j+1}(t_n)$ with

$$D(h_n^{j+1})T(a_n^{j+1} - 2b_n^{j+1}(h_n^{j+1})^2 t_n^{j+1})P(b_n^{j+1})U((h_n^{j+1})^2 t_n^{j+1} + s_n^{j+1})f$$

for suitable f . For simplicity, we drop upper index $j+1$ for a while. To this end, we shall recall that

$$v_n(t_n) = e^{i\theta_n} D(h_n)P(b_n)T(a_n - 2b_n(h_n)^2 t_n)\Phi((h_n)^2 t_n + s_n),$$

with suitable $\theta_n \in \mathbb{R}$. In view of (4.8), we may neglect $e^{i\theta_n}$. By extracting subsequence, we may suppose that $(h_n)^2 t_n + s_n$ converges to $T \in \overline{I_{\max}(\Phi)} \subset [-\infty, \infty]$. We first consider the case T is interior of $\overline{I_{\max}(\Phi)}$. In this case, $\Phi((h_n)^2 t_n + s_n)$ converges strongly to $\Phi(T)$ in $\hat{M}_{2,r}^{d\alpha}$. Hence, we may replace $\Phi((h_n)^2 t_n + s_n)$ by $U((h_n)^2 t_n + s_n)(U(-T)\Phi(T))$. Namely, we take $f = U(-T)\Phi(T)$. If $T = \sup I_{\max}(\Phi)$ then $T = \infty$ and Φ must scatters for positive time direction because t_n is taken from K_n^m . Hence, we may replace $\Phi((h_n)^2 t_n + s_n)$ by $U((h_n)^2 t_n + s_n)\Phi_+$ for some $\Phi_+ \in \hat{M}_{2,r}^{d\alpha}$. This implies that the choice $f = \Phi_+$ works. The case $T = \inf I_{\max}(\Phi)$ is handled similarly. Since $t_n \geq 0$ and $s_n \in I_{\max}(\Phi)$, this case occurs only if $s_n \rightarrow -\infty$ as $n \rightarrow \infty$ and $T = -\infty$. So, we may replace $\Phi((h_n)^2 t_n + s_n)$ by $U((h_n)^2 t_n + s_n)\Phi_-$ for some $\Phi_- \in \hat{M}_{2,r}^{d\alpha}$. The claim is true. Remark that

$$D(h_n)P(b_n)T(a_n - 2b_n(h_n)^2 t_n)U((h_n)^2 t_n + s_n)f = e^{i\gamma_n} U(t_n)\mathcal{G}_n f$$

for suitable $\gamma_n \in \mathbb{R}$.

We shall show

$$\sum_{\tau \in \mathcal{D}} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} |\langle \mathcal{F}f, \mathcal{F}r_n \rangle_\tau|^{\frac{r}{2}} \rightarrow 0$$

as $n \rightarrow \infty$, where

$$r_n := (\mathcal{G}_n^{j+1})^{-1} U(-t_n) \tilde{w}_n^{j+1}(t_n).$$

Since $f \in \hat{M}_{2,r}^{d\alpha}$ and r_n is uniformly bounded in $\hat{M}_{2,r}^{d\alpha}$, for any $\varepsilon > 0$ there exists a finite set of dyadic cubes $\Omega \subset \mathcal{D}$ independent of n such that

$$\begin{aligned} \sum_{\tau \in \mathcal{D} \setminus \Omega} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} |\langle \mathcal{F}f, \mathcal{F}r_n \rangle_\tau|^{\frac{r}{2}} &\leq \left(\sum_{\tau \in \mathcal{D} \setminus \Omega} |\tau|^{r(\frac{1}{2} - \frac{1}{d\alpha})} \|\mathcal{F}f\|_{L^2(\tau)}^r \right)^{\frac{1}{2}} \|r_n\|_{\hat{M}_{2,r}^{d\alpha}}^{\frac{r}{2}} \\ &\leq \varepsilon. \end{aligned}$$

Hence, the proof is reduced to showing that

$$\langle \mathcal{F}f, \mathcal{F}r_n \rangle_\tau \rightarrow 0$$

as $n \rightarrow \infty$ for each dyadic cube τ . A similar argument as in the previous paragraph allows us to replace \tilde{w}_n^{j+1} with

$$U(t_n) \left(\sum_{k=j+2}^J \mathcal{G}_n^k f^k + w_n^J \right).$$

With this replacement, it suffices to show

$$\left\langle \mathcal{F}f, \mathcal{F} \left(\sum_{k=j+2}^J (\mathcal{G}_n^{j+1})^{-1} \mathcal{G}_n^k f^k + (\mathcal{G}_n^{j+1})^{-1} w_n^J \right) \right\rangle_\tau \rightarrow 0$$

as $n \rightarrow \infty$. The desired convergence now follows from mutual orthogonality of families $\{\mathcal{G}_n^j\}_n \subset G$ ($j = 1, 2, 3, \dots$) and from weak-* convergence $(\mathcal{G}_n^{j+1})^{-1} w_n^J \rightharpoonup 0$ as $n \rightarrow \infty$ in $\hat{M}_{2,r}^{d\alpha}$. \square

Let us finish the proof of Proposition 4.2. So far, we obtain

$$u_n(0) = \mathcal{G}_n^1 \phi^1 + o(1)$$

in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$. Further, $s_n^1 \neq +\infty$ and the nonlinear profile $\Phi^1(t)$ does not scatter for positive time direction and so $\sup_{I_{\max}(\Phi^1) \cap \{t \geq 0\}} \ell_r(\Phi^1(t)) \geq E_c$. By the assumption (4.2), we see that

$$\sup_{I_{\max}(\Phi^1) \cap \{t \geq 0\}} \ell_r(\Phi^1(t)) = E_c$$

by the stability (or arguing as in Lemma 4.4).

Remark that we did not yet use the assumption $\lim_{n \rightarrow \infty} S_{\leq 0}(u_n) = \infty$. Arguing as in Lemmas 4.3 and 4.4, this assumption implies that $\Phi^1(t)$ have the same property for negative time direction. Namely, $s^1 \neq -\infty$, $\Phi^1(t)$ does not scatter for negative time direction, and

$$\sup_{I_{\max}(\Phi^1) \cap \{t \leq 0\}} \ell_r(\Phi^1(t)) = E_c.$$

We conclude that $\Phi^1(t)$ is the desired solution.

4.2. Almost periodicity modulo symmetry. In this section, we complete the proof of Theorem 4.1. By definition of E_c , we can take a sequence of solutions $u_n : I_n \times \mathbb{R}^d \rightarrow \mathbb{C}$ that satisfies the assumption of Proposition 4.2. Therefore, thanks to Proposition 4.2, we obtain a maximal-lifespan solution $v(t)$ that satisfies the first two properties of Theorem 4.1. Then, $E_2 = E_c$ follows. Thus, let us prove that $v(t)$ is almost periodic modulo symmetry as in (1.6). To this end, the main step is the following.

Proposition 4.5. *There exists $a(t) : I_{\max}(v) \rightarrow \mathbb{R}^d$, $b(t) : I_{\max}(v) \rightarrow \mathbb{R}^d$, $\lambda(t) : I_{\max}(v) \rightarrow 2^{\mathbb{Z}}$ such that the set*

$$\{(D(\lambda(t))P(b(t))T(a(t)))^{-1}v(t) \mid t \in I_{\max}(v)\} \subset \hat{M}_{2,r}^{d\alpha}$$

is totally bounded in $\hat{M}_{2,r}^{d\alpha}$.

Indeed, the property (1.6) for $v(t)$ then follows from the characterization of total boundedness (Theorem 2.19) with $N(t) = \lambda(t)$, $y(t) = a(t)/\lambda(t)$, and $z(t) = \lambda(t)b(t)$.

Remark 4.6. By using Proposition 4.2, we see that for any sequence $\{\tau_n\}_n \subset I_{\max}(v)$ there exists a sequence of parameters $(\lambda_n, a_n, b_n) \in 2^{\mathbb{Z}} \times \mathbb{R}^d \times \mathbb{R}^d$ such that $(D(\lambda_n)P(b_n)T(a_n))^{-1}v(\tau_n)$ possess a convergence subsequence. However, this statement is weaker. We have to choose parameters $a(t)$, $b(t)$, and $\lambda(t)$ independently of choice of a sequence $\{\tau_n\}_n$.

Proof. Step 1. We first construct $\lambda(t)$, $a(t)$, and $b(t)$. Fix $\sigma \in I_{\max}(v)$. For simplicity, we omit time variable σ and write $v = v(\sigma)$ in this step. Since $v(t)$ does not scatter for positive time direction, it holds that

$$(4.9) \quad S_{\mathbb{R}}(e^{it\Delta}v) \geq \delta,$$

where δ is the number given in Lemma 2.8. Moreover, we have

$$\|v\|_{\hat{M}_{2,r}^{d\alpha}} \leq 2^d E_c < \infty.$$

Mimicking the proof of Lemma B.5, we see that for any $\varepsilon > 0$ there exist a sequence of dyadic cubes $\{\tau_m\}_{m=1}^M \subset \mathcal{D}$, $M = M(\varepsilon)$, and constant $C_j = C_j(\varepsilon) > 0$ ($j = 1, 2$) such that if we define $f^m(x)$ by

$$\mathcal{F}f^m(\xi) := \mathcal{F}v(\xi) \times \mathbf{1}_{\tau_m \setminus (\cup_{k=1}^{m-1} \tau_k)}(\xi) \times \mathbf{1}_{\{|\mathcal{F}v(\xi)| \leq C_1 |\tau_m|^{-1/(d\alpha)'}\}}(\xi)$$

then it holds that

$$|\tau_m|^{\frac{1}{(d\alpha)'} - \frac{1}{2}} \|\mathcal{F}f^m\|_{L^2(\tau_m)} \geq C_2$$

for each $m = 1, 2, \dots, M$ and that

$$\left\| e^{it\Delta} \left(v - \sum_{m=1}^M f^m \right) \right\|_{L_{t,x}^{(d+2)\alpha}(\mathbb{R} \times \mathbb{R}^d)} \leq \varepsilon.$$

Choose $\varepsilon = \delta/3$. Then, at least one f^m satisfies

$$S_{\mathbb{R}}(e^{it\Delta} f^m) \geq \frac{\delta}{2M(\delta/3)} =: \delta_1.$$

Indeed, otherwise we obtain a contraction with (4.9). Pick such $m = m_0$ and define $\lambda(\sigma) \in 2^{\mathbb{Z}}$ and $b(\sigma) \in \mathbb{Z}^d$ by the relation

$$\tau_{m_0} = \lambda(\sigma)([0, 1]^d + b(\sigma)).$$

Set $g(x) = [P(b(\sigma))^{-1} D(\lambda(\sigma))^{-1} f^{m_0}](x)$. Then,

$$|\mathcal{F}g(\xi)| = (\lambda(\sigma))^{d-\frac{1}{\alpha}} |\mathcal{F}f^{m_0}(\lambda(\sigma)(\xi + b(\sigma)))| \leq C_1 \mathbf{1}_{[0,1]^d}(\xi).$$

Arguing as in [4, Lemma 22], we see that for any $\varepsilon > 0$ there exists a sequence of disjoint unit cubes $\{Q_k\}_{k=1}^K \subset \mathbb{R} \times \mathbb{R}^d$, $K = K(\varepsilon)$, such that

$$\|e^{it\Delta} g\|_{L_{t,x}^{(d+2)\alpha}((\mathbb{R} \times \mathbb{R}^d) \setminus \cup_{k=1}^K Q_k)} \leq \varepsilon.$$

Take $\varepsilon = \delta_1/3$. Then, at least for one k , we have

$$\|e^{it\Delta} g\|_{L_{t,x}^{(d+2)\alpha}(Q_k)} \geq \frac{\delta_1}{2K(\delta_1/3)} =: \delta_2$$

Choose such $k = k_0$ and define $a(\sigma)$ as the x -coordinate of the center of Q_{k_0} .

Step 2. We now prove that the above $\lambda(t)$, $a(t)$, and $b(t)$ give the desired conclusion. Take any sequence $\{t_n\} \subset I_{\max}(v)$, since the assumption of Proposition 4.2 is satisfied with $u_n \equiv v$ and this $\{t_n\}$, up to subsequence, we have

$$v(t_n) = D(\lambda_n)P(b_n)T(a_n)\phi + o(1)$$

in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$. Changing parameters and notations, and refining subsequence if necessary, we can rewrite the above convergence as

$$(4.10) \quad T(a(t_n))^{-1} P(b(t_n))^{-1} D(\lambda(t_n))^{-1} v(t_n) = T(a_n)^{-1} P(b_n)^{-1} D(\lambda_n)^{-1} \phi + o(1)$$

in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$. Without loss of generality, we may assume that $a_n, b_n \in \mathbb{Z}^d$. Now, it suffices to show that $|\log \lambda_n| + |a_n| + |b_n|$ is bounded (under the new representation).

Set $w(t) = T(a(t))^{-1} P(b(t))^{-1} D(\lambda(t))^{-1} v(t)$. By definition of $\lambda(t)$ and $b(t)$, we have

$$\|\mathbf{1}_{\{|\mathcal{F}w(t)| \leq C_1\}}(\mathcal{F}w(t))\|_{L^2([0,1]^d)} \geq C_2$$

for any $t \in I_{\max}(v)$. By (4.10), there exists n_0 such that

$$\begin{aligned} \frac{C_2}{2} &\leq \|\mathbf{1}_{\{|\mathcal{F}w(t_n)| \leq C_1\}} \mathcal{F}[T(a_n)^{-1} P(b_n)^{-1} D(\lambda_n)^{-1} \phi]\|_{L^2([0,1]^d)} \\ &\leq \|\mathcal{F}[T(a_n)^{-1} P(b_n)^{-1} D(\lambda_n)^{-1} \phi]\|_{L^2([0,1]^d)} \\ &= \lambda_n^{d(\frac{1}{2} - \frac{1}{d\alpha})} \|\mathcal{F}\phi\|_{L^2(\lambda_n([0,1]^d + b_n))} \end{aligned}$$

for $n \geq n_0$. Let N_0 be the number of the dyadic cubes $\tau \in \mathcal{D}$ such that

$$|\tau|^{\frac{1}{2} - \frac{1}{d\alpha}} \|\mathcal{F}\phi\|_{L^2(\tau)} \geq \frac{C_2}{2}.$$

Then, N_0 is bounded because

$$N_0 \left(\frac{C_2}{2} \right)^r \leq \|\phi\|_{\dot{M}_{2,r}^{d\alpha}}^r \leq (2^d E_c)^r.$$

Therefore, $\#\{\lambda_n([0,1]^d + b_n) \in \mathcal{D} \mid n \geq 1\} \leq n_0 + N_0 < \infty$. Hence, $\log |\lambda_n| + |b_n|$ is bounded.

Refining subsequence and changing notations, we may suppose that $\lambda_n \equiv 1$ and $b_n \equiv 0$. By definition of $a(t)$, we have

$$\|e^{it\Delta} \mathcal{F}^{-1}(\mathbf{1}_{A_n} \mathcal{F}w(t_n))\|_{L_{t,x}^{(d+2)\alpha}(\mathbb{R} \times [-1/2, 1/2]^d)} \geq \delta_2,$$

where $A_n \subset [0,1]^d$ is a suitable set depending only on $v(t_n)$. By (4.10) and Strichartz' estimate, there exists n_1 such that

$$\|e^{it\Delta} \mathcal{F}^{-1}(\mathbf{1}_{A_n} \mathcal{F}(T(a_n)^{-1} \phi))\|_{L_{t,x}^{(d+2)\alpha}(\mathbb{R} \times [-1/2, 1/2]^d)} \geq \frac{\delta_2}{2}$$

holds for all $n \geq n_1$. Let $N_1(n)$ be the number of vectors $a \in \mathbb{Z}^d$ such that

$$\|e^{it\Delta} \mathcal{F}^{-1}(\mathbf{1}_{A_n} \mathcal{F}\phi)\|_{L_{t,x}^{(d+2)\alpha}(\mathbb{R} \times ([-1/2, 1/2]^d + a))} \geq \frac{\delta_2}{2}.$$

Then, $N_1(n)$ is uniformly bounded because

$$\begin{aligned} N_1(n) \left(\frac{\delta_2}{2} \right)^{(d+2)\alpha} &\leq \|e^{it\Delta} \mathcal{F}^{-1}(\mathbf{1}_{A_n} \mathcal{F}\phi)\|_{L_{t,x}^{(d+2)\alpha}(\mathbb{R} \times \mathbb{R}^d)}^{(d+2)\alpha} \\ &\leq C \|\mathcal{F}^{-1}(\mathbf{1}_{A_n} \mathcal{F}\phi)\|_{\dot{M}_{2,r}^{d\alpha}}^{(d+2)\alpha} \\ &\leq C \|\phi\|_{\dot{M}_{2,r}^{d\alpha}}^{(d+2)\alpha} \leq C(2^d E_c)^{(d+2)\alpha}. \end{aligned}$$

Hence, $\#\{a_n \mid n \geq 1\} \leq n_1 + \sup_n N_1(n) < \infty$. In particular, a_n is bounded. \square

5. PROOF OF THEOREM 1.13

By definition of E_1 , we can take a sequence $u_{0,n} \in \dot{M}_{2,r}^{d\alpha}$ of initial data which satisfies the following two properties:

- (1) a solution $u_n(t)$ such that $u_n(0) = u_{0,n}$ does not scatter for positive time direction. Or equivalently,

$$(5.1) \quad S_{\geq 0}(u_n) = \infty.$$

(2) It has a size slightly bigger than E_1 i.e.

$$(5.2) \quad \ell_r(u_{0,n}) \leq E_1 + \frac{1}{n}.$$

We argue as in the proof of Proposition 4.2 to obtain the result. See [33, 34, 32] for details. We only comment the following two respects.

The first is that the case $s_n^j \rightarrow -\infty$ may happen. This difference comes from the fact that we do not necessarily have

$$\sup_n S_{\leq 0}(u_n) = \infty.$$

Remark that it is possible to choose a sequence $\{t_n\}_n$ so that a new sequence $\tilde{u}_{0,n} := u_n(t_n)$ satisfies this assumption and (5.1). However, we then lose the assumption (5.2) in general. In other words, a time translation argument is forbidden by the assumption (5.2). We also remark that if $s_n^1 \equiv 0$ then (2)-(a) of Theorem 1.13 occurs and if $s_n^1 \rightarrow -\infty$ then (2)-(b) of Theorem 1.13 takes place.

The second is that the proofs of $\phi^j \equiv 0$ for $j \geq 2$ and $w_n^1 \rightarrow 0$ in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$ are much simpler. By the linear profile decomposition, we obtain

$$u_{0,n} = \sum_{j=1}^J \mathcal{G}_n^j \phi^j + w_n^J$$

Then, the same argument as in the proof of Lemma 4.3 shows that one of $\Phi^j(t)$, a nonlinear profile associated with (ϕ^j, s_n^j) , does not scatter for positive time direction. Suppose that $\Phi^1(t)$ is the nonlinear profile does not scatter. Then, by definition of E_1 , we have $\ell_r(\phi^j) \geq E_1$. Combining the decoupling inequality, we immediately obtain the conclusion.

APPENDIX A. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2.

Proof. By [26, Theorem 1.7], we see that $E_2 = \|W\|_{\dot{H}^1}$ for $d \geq 5$. For $d = 4$, the same conclusion is obtained by [26, Theorem 1.16] and [7, Theorem 1.7].

Let us prove that $E_1 = \sqrt{2/d} \|W\|_{\dot{H}^1}$. We first remark that $\frac{1}{d} \|W\|_{\dot{H}^1}^2 = E[W]$. If a nontrivial $u_0 \in \dot{H}^1$ satisfies $\|u_0\|_{\dot{H}^1} \leq \sqrt{2/d} \|W\|_{\dot{H}^1}$ then it holds that

$$E[u_0] = \frac{1}{2} \|u_0\|_{\dot{H}^1}^2 - \frac{d-2}{2d} \|u_0\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} < \frac{1}{2} \|u_0\|_{\dot{H}^1}^2 \leq \frac{1}{d} \|W\|_{\dot{H}^1}^2 = E[W].$$

Thus, $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ and $E[u_0] < E[W]$ hold. Then, in light of [26, Corollary 1.9] and [7, Theorem 1.5], a corresponding solution $u(t)$ is global and scatters for both time directions. Hence, we obtain $E_1 \geq \sqrt{2/d} \|W\|_{\dot{H}^1}$.

On the other hand, [10, Theorem 1] and [30, Theorem 1.3 and Proposition 1.5] show that there exists a radial global solution $W_-(t)$ such that

- (1) $E[W_-] = E[W]$;
- (2) $W_-(t)$ scatters for negative time direction;
- (3) $W_-(t)$ converges to W exponentially as $t \rightarrow \infty$, that is, there exists positive constants c and C such that

$$\|W_-(t) - W\|_{\dot{H}^1(\mathbb{R}^d)} \leq C e^{-ct}$$

for all $t \geq 0$. In particular, $\lim_{t \rightarrow \infty} \|W_-(t)\|_{\dot{H}^1} = \|W\|_{\dot{H}^1} = E_2$.
 for $d \geq 3$. Since $W_-(t)$ scatters for negative time direction, we have $\|W_-(t)\|_{L^{\frac{2d}{d-2}}} \rightarrow 0$ as $t \rightarrow -\infty$. Combining with the energy conservation, we deduce that

$$\frac{1}{2} \|W_-(t)\|_{\dot{H}^1}^2 = E[W] + \frac{d-2}{2d} \|W_-(t)\|_{L^{\frac{2d}{d-2}}}^{\frac{2d}{d-2}} \rightarrow E[W]$$

as $t \rightarrow -\infty$. Namely,

$$\|W_-(t)\|_{\dot{H}^1} \rightarrow \sqrt{2E[W]} = \sqrt{\frac{2}{d}} \|W\|_{\dot{H}^1}$$

as $t \rightarrow -\infty$. Since $\|W_-(t)\|_{\dot{H}^1}$ is continuous in time, this shows

$$E_1 \leq \lim_{t \rightarrow -\infty} \|W_-(t)\|_{\dot{H}^1} = \sqrt{\frac{2}{d}} \|W\|_{\dot{H}^1},$$

which completes the proof. \square

Remark A.1. If we consider the problem under the radial symmetry then we obtain the same conclusion for $d = 3$ by using a result by Kenig and Merle [22]. Without the radial symmetry, we only have the upper bounds $E_1 \leq \sqrt{2/d} \|W\|_{\dot{H}^1}$ and $E_2 \leq \|W\|_{\dot{H}^1}$.

APPENDIX B. PROOF OF THEOREM 3.11

In this section, we prove a linear profile decomposition. Throughout this section, we assume that $d \geq 1$ and

$$\frac{2}{d} \cdot \frac{1}{1 + \frac{2}{d(d+3)}} < \alpha < \frac{2}{d}, \quad (d\alpha)' < r < ((d+2)\alpha)^*.$$

The proof consists of two parts. The first is a decomposition of a bounded sequence of functions in $\hat{M}_{2,r}^{d\alpha}$ with a different notion of smallness of remainders. The second is a concentration compactness type result, which assures that the modified notion of smallness is stronger than the original one.

B.1. Decomposition of a sequence. Let us first introduce notations. For a bounded sequence $P = \{P_n\}_n \subset \hat{M}_{2,r}^{d\alpha}$, we introduce a *set of weak-* limits modulo deformations*

$$\mathcal{M}(P) := \left\{ \phi \in \hat{M}_{2,r}^{d\alpha} \left| \begin{array}{l} \phi = \lim_{k \rightarrow \infty} \mathcal{G}_{n_k}^{-1} P_{n_k} \text{ weakly-} * \text{ in } \hat{M}_{2,r}^{d\alpha}, \\ \exists \mathcal{G}_n \in G, \exists \text{subsequence } n_k \end{array} \right. \right\}.$$

and define

$$\eta(P) := \sup_{\phi \in \mathcal{M}(P)} \ell_r(\phi),$$

where $\ell_r(\cdot)$ is the size function introduced in (1.5). The main result of this section is a decomposition with a smallness of remainders with respect to η .

Theorem B.1. *Let $u = \{u_n\}_n$ be a bounded sequence of functions in $\hat{M}_{2,r}^{d\alpha}$. Then, there exist $\phi^j \in \mathcal{M}(u)$, $R_n^l \in \hat{M}_{2,r}^{d\alpha}$, and pairwise orthogonal families of deformations $\{\mathcal{G}_n^j\}_n \subset G$ ($j = 1, 2, \dots$) such that, up to subsequence, a*

decomposition (3.3) holds for any $l, n \geq 1$. Moreover, $\{R_n^j\}_{n,j}$ satisfies the convergence (3.4) and

$$(B.1) \quad \eta(R^l) \rightarrow 0$$

as $l \rightarrow \infty$. Furthermore, a decoupling inequality (3.6) holds for any $J \geq 1$.

We first recall a decoupling inequality in [32].

Lemma B.2 (Decoupling inequality). *Let $\{u_n\}_n$ be a bounded sequence in $\hat{M}_{2,r}^{d\alpha}$. Let $\{\mathcal{G}_n\}_n \subset G$ be a sequence of deformations. Suppose that $\mathcal{G}_n^{-1}u_n$ converges to ϕ weakly- $*$ in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$. Set $R_n := u_n - \mathcal{G}_n\phi$. Then, for any $\gamma > 1$ and $b_0 \in \mathbb{R}^d$, it holds that*

$$(B.2) \quad \gamma \|P(b_0)u_n\|_{\hat{M}_{2,r}^{d\alpha}}^r \geq \|P(b_0)\mathcal{G}_n\phi\|_{\hat{M}_{2,r}^{d\alpha}}^r + \|P(b_0)R_n\|_{\hat{M}_{2,r}^{d\alpha}}^r + o_\gamma(1)$$

as $n \rightarrow \infty$.

The idea of the proof is to sum up the local (in the Fourier side) L^2 decoupling with respect to intervals. We do not repeat details.

Proof of Theorem B.1. We may suppose $\eta(u) > 0$, otherwise the result holds with $\phi^j \equiv 0$ and $R_n^j = u_n$ for all $j \geq 1$. Then, we can choose $\phi^1 \in \mathcal{M}(u)$ so that $\ell_r(\phi^1) \geq \frac{1}{2}\eta(u)$ by definition of η . Then, by definition of $\mathcal{M}(u)$, one finds $\mathcal{G}_n^1 \in G$ such that

$$(\mathcal{G}_n^1)^{-1}u_n \rightharpoonup \phi^1 \quad \text{weakly-} * \quad \text{in } \hat{M}_{2,r}^{d\alpha}$$

as $n \rightarrow \infty$ up to subsequence. Define $R_n^1 := u_n - \mathcal{G}_n^1\phi^1$. Then, (3.3) holds for $l = 1$. It is obvious that

$$(B.3) \quad (\mathcal{G}_n^1)^{-1}R_n^1 \rightharpoonup \phi^1 - \phi^1 = 0 \quad \text{weakly-} * \quad \text{in } \hat{M}_{2,r}^{d\alpha}$$

as $n \rightarrow \infty$. By Lemma B.2,

$$(B.4) \quad \gamma \|P(b_0)u_n\|_{\hat{M}_{2,r}^{d\alpha}}^r \geq \|P(b_0)\mathcal{G}_n^1\phi^1\|_{\hat{M}_{2,r}^{d\alpha}}^r + \|P(b_0)R_n^1\|_{\hat{M}_{2,r}^{d\alpha}}^r + o_\gamma(1)$$

as $n \rightarrow \infty$ for any constant $\gamma > 1$ and $b_0 \in \mathbb{R}^d$. Since $\gamma > 1$ and b_0 are arbitrary, the decoupling inequality (3.6) holds for $J = 1$.

If $\eta(R^1) = 0$ then the proof is completed by taking $\phi^j \equiv 0$ for $j \geq 2$. Otherwise, we can choose $\phi^2 \in \mathcal{M}(R^1)$ so that $\ell_r(\phi^2) \geq \frac{1}{2}\eta(R^1)$. Then, as in the previous step, one can take $\mathcal{G}_n^2 \in G$ so that

$$(\mathcal{G}_n^2)^{-1}R_n^1 \rightharpoonup \phi^2 \quad \text{weakly-} * \quad \text{in } \hat{M}_{2,r}^{d\alpha}$$

as $n \rightarrow \infty$ up to subsequence. In particular, $\phi^2 \neq 0$. Together with (B.3), Proposition 3.8 (3) gives us that $\{(\mathcal{G}_n^2)^{-1}\mathcal{G}_n^1\}_n$ is vanishing. Hence, \mathcal{G}_n^1 and \mathcal{G}_n^2 are orthogonal. Then, Proposition 3.8 (2) implies that

$$(\mathcal{G}_n^2)^{-1}u_n = (\mathcal{G}_n^2)^{-1}\mathcal{G}_n^1\phi^1 + (\mathcal{G}_n^2)^{-1}R_n^1 \rightarrow 0 + \phi^2 \quad \text{weakly-} * \quad \text{in } \hat{M}_{2,r}^{d\alpha}$$

as $n \rightarrow \infty$. Hence, we obtain $\phi^2 \in \mathcal{M}(u)$. Set $R_n^2 := R_n^1 - \mathcal{G}_n^2\phi^2$. Then, (3.3) holds for $l = 2$. Further, one deduces from Lemma B.2 that

$$\gamma \|P(b_0)R_n^1\|_{\hat{M}_{2,r}^{d\alpha}}^r \geq \|P(b_0)\mathcal{G}_n^2\phi^2\|_{\hat{M}_{2,r}^{d\alpha}}^r + \|P(b_0)R_n^2\|_{\hat{M}_{2,r}^{d\alpha}}^r + o_\gamma(1)$$

as $n \rightarrow \infty$ for any $\gamma > 1$ and $b_0 \in \mathbb{R}^d$. This implies (3.6) for $J = 2$ with the help of (B.4).

Repeat this argument and construct $\phi^j \in \mathcal{M}(R^{j-1})$ and $\mathcal{G}_n^j \in G$, inductively. If we have $\eta(R^{j_0}) = 0$ for some j_0 then, for $j \geq j_0 + 1$, we take $\phi^j \equiv 0$ and define suitable \mathcal{G}_n^j so that mutual orthogonality holds. In what follows, we may suppose that $\eta(R^j) > 0$ for all $j \geq 1$. In each step, R_n^j is defined by the formula $R_n^j = R_n^{j-1} - \mathcal{G}_n^j \phi^j$. The decomposition (3.3) is obvious by construction.

Let us now prove that pairwise orthogonality. To this end, we demonstrate by induction that \mathcal{G}_n^j is orthogonal to \mathcal{G}_n^k for $1 \leq k \leq j-1$. Since $(\mathcal{G}_n^j)^{-1} r_n^j \rightharpoonup \phi^j$ and $(\mathcal{G}_n^{j-1})^{-1} r_n^{j-1} \rightharpoonup 0$ weakly-* in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$, Proposition 3.8 implies that \mathcal{G}_n^j and \mathcal{G}_n^{j-1} are orthogonal. Suppose that \mathcal{G}_n^j is orthogonal to \mathcal{G}_n^k for $k_0 \leq k \leq j-1$. Proposition 3.8 (2) yields

$$(\mathcal{G}_n^j)^{-1} R_n^{k_0-1} = \sum_{k=k_0}^{j-1} (\mathcal{G}_n^j)^{-1} \mathcal{G}_n^k \phi^k + (\mathcal{G}_n^j)^{-1} R_n^{j-1} \rightharpoonup \phi^j \quad \text{weakly-* in } \hat{M}_{2,r}^{d\alpha}$$

as $n \rightarrow \infty$. On the other hand, $(\mathcal{G}_n^{k_0-1})^{-1} R_n^{k_0-1} \rightharpoonup 0$ as $n \rightarrow \infty$. We therefore see that \mathcal{G}_n^j and $\mathcal{G}_n^{k_0-1}$ are orthogonal. Hence, by induction, \mathcal{G}_n^j is orthogonal to \mathcal{G}_n^k for $1 \leq k \leq j-1$.

The above argument also proves that the convergence (3.4). and $\phi^j \in \mathcal{M}(u)$ for all $j \geq 1$.

To conclude the proof, we shall show (B.1) and (3.6). Notice that the inductive construction gives us

$$(B.5) \quad \ell_r(\phi^{j+1}) \geq \frac{1}{2} \eta(R^j)$$

for $j \geq 1$ and

$$(B.6) \quad \gamma \|P(b_0) R_n^j\|_{\hat{M}_{2,r}^{d\alpha}}^r \geq \|P(b_0) \mathcal{G}_n^{j+1} \phi^{j+1}\|_{\hat{M}_{2,r}^{d\alpha}}^r + \|P(b_0) R_n^{j+1}\|_{\hat{M}_{2,r}^{d\alpha}}^r + o_{\gamma,j}(1).$$

as $n \rightarrow \infty$ for (fixed) $j \geq 1$ and any $\gamma > 1$ and $b_0 \in \mathbb{R}^d$. Combining (B.4) and (B.6) for $1 \leq j \leq J$, we have

$$\begin{aligned} \gamma^J \|P(b_0) u_n\|_{\hat{M}_{2,r}^{d\alpha}}^r &\geq \sum_{j=1}^J \gamma^{J-j} \|P(b_0) \mathcal{G}_n^j \phi^j\|_{\hat{M}_{2,r}^{d\alpha}}^r + \|P(b_0) R_n^J\|_{\hat{M}_{2,r}^{d\alpha}}^r + o_{\gamma,J}(1) \\ &\geq \sum_{j=1}^J \gamma^{J-j} \ell_r(\phi^j)^r + \ell_r(R_n^J)^r + o_{\gamma,J}(1). \end{aligned}$$

Take first infimum with respect to b_0 and then limit supremum in n to obtain

$$\limsup_{n \rightarrow \infty} \ell_r(u_n)^r \geq \sum_{j=1}^J \gamma^{-j} \ell_r(\phi^j)^r + \gamma^{-J} \limsup_{n \rightarrow \infty} \ell_r(R_n^J)^r.$$

Since $\gamma > 1$ is arbitrary, we obtain (3.6). Finally, (3.6) and (B.5) imply (B.1). \square

B.2. Concentration compactness. The second part of the proof of Theorem 3.11 is concentration compactness. Intuitively, the meaning of the concentration compactness is as follows. Let us consider a bonded sequence $\{u_n\}_n \subset \hat{M}_{2,r}^{d\alpha}$. Without any additional assumption, we may not expect to find any *nonzero* weak-* limit of the sequence. Such a sequence is easily constructed by considering an orbit of *general deformations*². So, to find a nonzero limit, we make some additional assumption on the sequence. If the additional assumption is so strong that it removes all possible deformations that $\{u_n\}_n$ may possess with few exceptions, say G , then we can find a nonzero weak-* limit modulo G . In our case, the additional assumption is (B.8) below. Recall that $S_I(u) := \|u\|_{L_{t,x}^{(d+2)\alpha}(I \times \mathbb{R}^d)}$.

Theorem B.3 (Concentration compactness). *Let a bounded sequence $\{u_n\} \subset \hat{M}_{2,r}^{d\alpha}$ satisfy*

$$(B.7) \quad \|u_n\|_{\hat{M}_{2,r}^{d\alpha}} \leq M$$

and

$$(B.8) \quad S_{\mathbb{R}}(e^{it\Delta}u_n) \geq m$$

for some positive constants m, M . Then, $\eta(u) \geq \beta(m, M)$ holds for some positive constant $\beta(m, M)$ depending only on m, M .

Remark B.4. The choice of a set G given in Definition 3.3 is the best one. Remark that the assumptions (B.7) and (B.8) are “preserved” under $D(h), T(a), P(b)$, and $U(s)$, and so we cannot remove any of these actions from G . The main point of the above theorem is that a set G with these four actions is “enough.”

Plugging Theorem B.3 to Theorem B.1, we obtain desired decomposition result. Before proceeding to the proof of Theorem B.3, we complete the proof Theorem 3.11.

Proof of Theorem 3.11. By means of Theorem B.1, it suffices to show (3.5) as $l \rightarrow \infty$. Assume for contradiction that a sequence R_n^l given in Theorem B.1 satisfies

$$\limsup_{l \rightarrow \infty} \limsup_{n \rightarrow \infty} S_{\mathbb{R}}(e^{it\Delta}R_n^l) > 0.$$

Then, we can choose $m > 0$ and a subsequence l_k with $l_k \rightarrow \infty$ as $k \rightarrow \infty$ such that the assumption of Theorem B.3 is fulfilled for each k . Then, Theorem B.3 implies $\eta(R^{l_k}) \geq C\beta > 0$, which contradicts to (B.1). \square

The proof of Theorem B.3 consists of three steps. The argument is very close to that in the mass-critical case $\alpha = 2$ such as [35, 6, 3] or that for generalized KdV equation [45, 32]. Throughout the proof, we fix \tilde{q} and \tilde{r} so that

$$2 < \tilde{q} < \left(d + \frac{2}{d+3}\right)\alpha, \quad \tilde{r} = ((d+2)\alpha)^*.$$

² Here, general deformation means a set of bounded linear operators on $\hat{M}_{2,r}^{d\alpha}$ that the norms of itself and its inverse are uniformly bounded. The set G given in Definition 3.3 is an example. There exist infinitely many such a set in view of multiplier type operators $\{e^{it\phi(i\nabla)}\}_{t \in \mathbb{R}}$, where ϕ is a real valued function. The notion of general deformation is in the same spirit as what is called as *displacement* in [44].

Remark that this choice gives

$$(B.9) \quad cS_{\mathbb{R}}(e^{it\Delta}f) \leq \|f\|_{\hat{M}_{q,\tilde{r}}^{d\alpha}} \leq \|f\|_{\hat{M}_{2,r}^{d\alpha}}$$

by Proposition 2.1, where c is a positive constant.

Step 1 – Decomposition into a sum of scale pieces. Let us start the proof of Theorem B.3 with a decomposition of bounded sequence into some pieces of which Fourier transforms have mutually disjoint compact supports and are bounded.

Lemma B.5. *Suppose that a sequence $\{u_n\}_n \subset \hat{M}_{2,r}^{d\alpha}$ satisfy $\|u_n\|_{\hat{M}_{2,r}^{d\alpha}} \leq M$. Then, for any $\varepsilon > 0$, there exist a subsequence of n which denoted still by n , a number J , a sequence of dyadic cubes $\{\tau_n^j = h_n^j([0,1]^d + b_n^j)\}_n \subset \mathcal{D}$ ($h_n^j \in 2^{\mathbb{Z}}$, $b_n^j \in \mathbb{Z}^d$, $1 \leq j \leq J$), $\{f_n^j\}_n \subset \hat{M}_{2,r}^{d\alpha}$ ($1 \leq j \leq J$), and $q_n \in \hat{M}_{2,r}^{d\alpha}$ such that*

$$\left| \log \frac{h_n^j}{h_n^k} \right| + \left| b_n^j - \frac{h_n^k}{h_n^j} b_n^k \right| \rightarrow \infty$$

as $n \rightarrow \infty$ for $1 \leq j < k \leq J$, and u_n is decomposed into

$$(B.10) \quad u_n = \sum_{j=1}^J f_n^j + q_n$$

for all $n \geq 1$. Moreover, it holds that

$$\|u_n\|_{\hat{M}_{2,r}^{d\alpha}}^r \geq \sum_{j=1}^J \|f_n^j\|_{\hat{M}_{2,r}^{d\alpha}}^r + \|q_n\|_{\hat{M}_{2,r}^{d\alpha}}^r$$

for all $n \geq 1$ and

$$(B.11) \quad \limsup_{n \rightarrow \infty} \|q_n\|_{\hat{M}_{q,\tilde{r}}^{d\alpha}} \leq \varepsilon.$$

Further, there exists a bounded and compactly supported function F_j such that $\widehat{f_n^j}$ satisfies

$$(B.12) \quad |\tau_n^j|^{\frac{1}{\alpha'}} |\mathcal{F} f_n^j(h_n^j(\xi + b_n^j))| \leq F_j(\xi)$$

for any $n \geq 1$.

Proof. Pick $\varepsilon > 0$. If $\limsup_{n \rightarrow \infty} \|u_n\|_{\hat{M}_{q,\tilde{r}}^{d\alpha}} \leq \varepsilon$ then there is nothing to be proved. Otherwise, we can extract a subsequence so that

$$\|u_n\|_{\hat{M}_{q,\tilde{r}}^{d\alpha}} > \varepsilon$$

for all n . By Hölder's inequality, the embedding $\hat{M}_{2,r}^{d\alpha} \hookrightarrow \hat{M}_{q,r}^{d\alpha}$ ($q > 2$), and assumption, one sees that

$$\varepsilon \leq \|u_n\|_{\hat{M}_{q,r}^{d\alpha}}^{1-\theta} \|u_n\|_{\hat{M}_{q,\infty}^{d\alpha}}^{\theta} \leq \|u_n\|_{\hat{M}_{2,r}^{d\alpha}}^{1-\theta} \|u_n\|_{\hat{M}_{q,\infty}^{d\alpha}}^{\theta} \leq M^{1-\theta} \|u_n\|_{\hat{M}_{q,\infty}^{d\alpha}}^{\theta},$$

where $\theta = \theta(r, \tilde{r}) \in (0, 1)$. Hence, there exists a dyadic cube $\tau_n^1 := h_n^1([0,1]^d + b_n^1) \in \mathcal{D}$ with $h_n^1 \in 2^{\mathbb{Z}}$ and $b_n^1 \in \mathbb{Z}^d$ such that

$$(B.13) \quad \int_{\tau_n^1} |\hat{u}_n|^{\tilde{q}'} d\xi \geq C_1 \varepsilon^{\frac{\tilde{q}'}{\theta}} |\tau_n^1|^{\left(\frac{1}{\tilde{q}'} - \frac{1}{(d\alpha)'}\right)\tilde{q}'},$$

where $C_1 = C_1(r, \tilde{r}, M)$ is a positive constant. On the other hand, it holds for any $A > 0$ that

$$\begin{aligned}
 \int_{\tau_n^1 \cap \{|\hat{u}_n| \geq A\}} |\hat{u}_n|^{\tilde{q}'} d\xi &\leq A^{\tilde{q}'-2} \|\hat{u}_n\|_{L^2(\tau_n^1)}^2 \\
 &\leq A^{\tilde{q}'-2} |\tau_n^1|^{2(\frac{1}{d\alpha}-\frac{1}{2})} \|u_n\|_{\hat{M}_{2,r}^{d\alpha}}^2 \\
 &\leq M^2 A^{\tilde{q}'-2} |\tau_n^1|^{\frac{2}{d\alpha}-1}.
 \end{aligned}
 \tag{B.14}$$

since $\tilde{q}' < 2$. We choose A so that

$$M^2 A^{\tilde{q}'-2} |\tau_n^1|^{\frac{2}{d\alpha}-1} = \frac{C_1}{2} \varepsilon^{\frac{\tilde{q}'}{\theta}} |\tau_n^1|^{(\frac{1}{\tilde{q}'} - \frac{1}{(d\alpha)'})\tilde{q}'},$$

or more explicitly,

$$A = \left(\frac{2}{C_1 M^2} \right)^{\frac{1}{2-\tilde{q}'}} \varepsilon^{\frac{1}{\theta} \left(\frac{\tilde{q}'}{2-\tilde{q}'} \right)} |\tau_n^1|^{-\frac{1}{(d\alpha)'}} =: C_\varepsilon |\tau_n^1|^{-\frac{1}{(d\alpha)'}}.$$

From (B.13) and (B.14), we have

$$\int_{\tau_n^1 \cap \{|\hat{u}_n| \leq C_\varepsilon |\tau_n^1|^{-1/(d\alpha)'}\}} |\hat{u}_n|^{\tilde{q}'} d\xi \geq \frac{C_1}{2} \varepsilon^{\frac{\tilde{q}'}{\theta}} |\tau_n^1|^{(\frac{1}{\tilde{q}'} - \frac{1}{(d\alpha)'})\tilde{q}'}.
 \tag{B.15}$$

Hölder's inequality implies that

$$\begin{aligned}
 \int_{\tau_n^1 \cap \{|\hat{u}_n| \leq C_\varepsilon |\tau_n^1|^{-1/(d\alpha)'}\}} |\hat{u}_n|^{\tilde{q}'} d\xi \\
 \leq \left(\int_{\tau_n^1 \cap \{|\hat{u}_n| \leq C_\varepsilon |\tau_n^1|^{-1/(d\alpha)'}\}} |\hat{u}_n|^2 d\xi \right)^{\frac{\tilde{q}'}{2}} |\tau_n^1|^{1-\frac{\tilde{q}'}{2}}.
 \end{aligned}
 \tag{B.16}$$

Combining (B.15) and (B.16), we reach to the estimate

$$|\tau_n^1|^{\frac{1}{(d\alpha)'}-\frac{1}{2}} \left(\int_{\tau_n^1 \cap \{|\hat{u}_n| \leq C_\varepsilon |\tau_n^1|^{-1/(d\alpha)'}\}} |\hat{u}_n|^2 d\xi \right)^{\frac{1}{2}} \geq \left(\frac{C_1}{2} \right)^{\frac{1}{\tilde{q}'}} \varepsilon^{\frac{1}{\theta}}.
 \tag{B.17}$$

We define v_n^1 by $\widehat{v_n^1} := \hat{u}_n \mathbf{1}_{\tau_n^1 \cap \{|\hat{u}_n| \leq C_\varepsilon |\tau_n^1|^{-1/(d\alpha)'}\}}$ and $q_n^1 := u_n - v_n^1$. Then, (B.17) implies that $\|v_n^1\|_{\hat{M}_{2,r}^{d\alpha}} \geq C\varepsilon^{\frac{1}{\theta}}$. Further, we have

$$|\tau_n^1|^{\frac{1}{(d\alpha)'}} \left| \widehat{v_n^1}(h_n^1(\xi + b_n^1)) \right| \leq C_\varepsilon \mathbf{1}_{[0,1]^d}(\xi).$$

If $\limsup_{n \rightarrow \infty} \|q_n^1\|_{\hat{M}_{q,r}^{d\alpha}} \leq \varepsilon$ then we have done. Otherwise, the same argument with u_n being replaced by q_n^1 enables us to define $\tau_n^2 := h_n^2([0,1]^d + b_n^2)$, v_n^2 , and q_n^2 (up to subsequence). We repeat this argument and define $\tau_n^j := h_n^j([0,1]^d + b_n^j)$, v_n^j , and q_n^j inductively. It is easy to see that

$$\|u_n\|_{\hat{M}_{2,r}^{d\alpha}}^r \geq \sum_{j=1}^N \|v_n^j\|_{\hat{M}_{2,r}^{d\alpha}}^r + \|q_n^N\|_{\hat{M}_{2,r}^{d\alpha}}^r$$

since supports of $\{v_n^j\}_{1 \leq j \leq N}$ and q_n^N are disjoint in the Fourier side and since $r > 2$. Since $\|v_n^j\|_{\hat{M}_{2,r}^{d\alpha}} \geq C\varepsilon^{\frac{1}{\theta}}$ for each j , together with (B.9), we see that

$$\limsup_{n \rightarrow \infty} \|q_n^J\|_{\hat{M}_{\tilde{q},\tilde{r}}^{d\alpha}} \leq \varepsilon$$

holds in at most $J = J(\varepsilon)$ steps. Set $q_n := q_n^J$.

We reorganize v_n^j to obtain mutual asymptotic orthogonality. It is done as follows; We collect all $k \geq 2$ such that $|\log \frac{h_n^1}{h_n^k}| + \left|b_n^1 - \frac{h_n^k}{h_n^1} b_n^k\right|$ is bounded, and define $f_n^1 := v_n^1 + \sum_k v_n^k$. Since

$$\begin{aligned} & |\tau_n^1|^{\frac{1}{(d\alpha)'}} \left| \widehat{v_n^k}(h_n^1(\xi + b_n^1)) \right| \\ &= \left(\frac{h_n^1}{h_n^k} \right)^{\frac{1}{(d\alpha)'}} |\tau_n^k|^{\frac{1}{(d\alpha)'}} \left| \widehat{v_n^k} \left(h_n^k \left[\frac{h_n^1}{h_n^k} \left\{ \xi + \left(b_n^1 - \frac{h_n^k}{h_n^1} b_n^k \right) \right\} + b_n^k \right] \right) \right| \\ &\leq C_\varepsilon \left(\frac{h_n^1}{h_n^k} \right)^{\frac{1}{(d\alpha)'}} \mathbf{1}_{[0,1]^d} \left(\frac{h_n^1}{h_n^k} \left\{ \xi + \left(b_n^1 - \frac{h_n^k}{h_n^1} b_n^k \right) \right\} \right), \end{aligned}$$

we see that $|\tau_n^1|^{\frac{1}{(d\alpha)'}} \mathcal{F} f_n^1(h_n^1(\xi + b_n^1)) \leq F_1(\xi)$ for some bounded and compactly supported function F_1 . Similarly, we define f_n^j inductively. It is easy to see that f_n^j possesses all properties we want. This completes the proof of Lemma B.5. \square

Step 2 – Decomposition of each scale pieces. We next decompose functions obtained in the previous decomposition.

Lemma B.6. *Let $F(\xi)$ be a nonnegative bounded function with compact support. Suppose that a sequence $R_n \in \hat{M}_{2,r}^{d\alpha}$ satisfy*

$$(B.18) \quad |\widehat{R_n}(\xi)| \leq F(\xi).$$

Then, up to subsequence, there exist $\{\phi^l\}_l \subset \hat{M}_{2,r}^{d\alpha}$ with $|\widehat{\phi^l}(\xi)| \leq F(\xi)$, $(a_n^l, s_n^l) \in \mathbb{R}^d \times \mathbb{R}^d$ with

$$\lim_{n \rightarrow \infty} (|s_n^l - s_n^{\tilde{l}}| + |a_n^l - a_n^{\tilde{l}}|) = \infty$$

for any $\tilde{l} \neq l$, and $\{r_n^l\}_{n,l} \subset \hat{M}_{2,r}^{d\alpha}$ with $|\widehat{r_n^l}(\xi)| \leq F(\xi)$ such that

$$R_n(x) = \sum_{l=1}^L U(s_n^l) T(a_n^l) \phi^l(x) + r_n^L(x)$$

for any $L \geq 1$. Moreover, it holds that

$$(B.19) \quad \sum_{l=1}^L \left\| \phi^l \right\|_{\hat{M}_{2,r}^q}^r + \limsup_{n \rightarrow \infty} \|r_n^L\|_{\hat{M}_{2,r}^q}^r \leq \limsup_{n \rightarrow \infty} \|R_n\|_{\hat{M}_{2,r}^q}^r < \infty$$

for any $2 < q' < r < \infty$ and $L \geq 1$. Furthermore, we have

$$(B.20) \quad \limsup_{n \rightarrow \infty} S_{\mathbb{R}}(e^{it\Delta} r_n^L) \rightarrow 0$$

as $L \rightarrow \infty$.

For the proof, see [6, Proposition 3.4]. A key restriction estimate in our case is established in Proposition 2.1.

Step 3 – Completion of the proof of Theorem B.3. We are now ready to prove Theorem B.3. For the proof, we recall the following space-time nonresonant property.

Lemma B.7. *Let $\phi^j \in \hat{M}_{2,r}^{d\alpha}$ ($1 \leq j \leq J$). Let $\{\mathcal{G}_n^j\}_n \subset G$ ($1 \leq j \leq J$) be mutually orthogonal families. Then,*

$$\left\| \sum_{j=1}^J e^{it\Delta} \mathcal{G}_n^j \phi^j \right\|_{L_{t,x}^{(d+2)\alpha}(\mathbb{R} \times \mathbb{R}^d)}^{(d+2)\alpha} \leq \sum_{j=1}^J \left\| e^{it\Delta} \mathcal{G}_n^j \phi^j \right\|_{L_{t,x}^{(d+2)\alpha}(\mathbb{R} \times \mathbb{R}^d)}^{(d+2)\alpha} + o(1)$$

as $n \rightarrow \infty$.

The proof is standard. For instance, see [3, Lemma 5.5]. Note that $(d+2)\alpha > \frac{2(d+3)}{d+1} > 2$.

Proof of Theorem B.3. Let $\{u_n\} \subset \hat{M}_{2,r}^{d\alpha}$ be a bounded sequence satisfying (B.7) and (B.8). Let $\varepsilon = \varepsilon(m, M) > 0$ to be chosen later. Let $J = J(\varepsilon) \geq 1$, $\{I_n^j = h_n^j([0, 1]^d + b_n^j)\}_n \subset \mathcal{D}$ ($1 \leq j \leq J$), $\{f_n^j\}_n \subset \hat{M}_{2,r}^{d\alpha}$ ($1 \leq j \leq J$), and q_n be sequences given in Lemma B.5. Set

$$\widehat{R_n^j}(\xi) := |h_n^j|^{\frac{d}{(d\alpha)'}} \widehat{f_n^j}(h_n^j(\xi + b_n^j)).$$

Namely, $R_n^j = P(b_n^j)^{-1} D(h_n^j)^{-1} f_n^j$. Then, by means of (B.12), $\{R_n^j\}_n$ satisfies assumption of Lemma B.6 for each j . Then, thanks to Lemma B.6, for every $1 \leq j \leq J$, there exists a family $\{\phi^{j,l}\}_l \subset \hat{M}_{2,r}^{d\alpha}$, and a family $\{(a_n^{j,l}, s_n^{j,l})\}_{n,l} \in \mathbb{R}^d \times \mathbb{R}^d$ such that

$$R_n^j = \sum_{l=1}^L U(s_n^{j,l}) T(a_n^{j,l}) \phi^{j,l} + r_n^{j,L}$$

with

$$\limsup_{n \rightarrow \infty} S_{\mathbb{R}}(e^{it\Delta} r_n^{j,L}) \rightarrow 0$$

as $L \rightarrow \infty$ and that

$$\lim_{n \rightarrow \infty} (|s_n^{j,l} - s_n^{j,\tilde{l}}| + |a_n^{j,l} - a_n^{j,\tilde{l}}|) = \infty$$

for any $l \neq \tilde{l}$. Remark that

$$\begin{aligned} \text{(B.21)} \quad f_n^j &= D(h_n^j) P(b_n^j) R_n^j \\ &= \sum_{l=1}^L D(h_n^j) P(b_n^j) U(s_n^{j,l}) T(a_n^{j,l}) \phi^{j,l} + D(h_n^j) P(b_n^j) r_n^{j,L}. \end{aligned}$$

We choose $L = L(\varepsilon)$ so that

$$\text{(B.22)} \quad \limsup_{n \rightarrow \infty} S_{\mathbb{R}}(e^{it\Delta} D(h_n^j) P(b_n^j) r_n^{j,L}) \leq \frac{\varepsilon}{J}$$

holds for any $1 \leq j \leq J$. Notice that this is possible by means of the scale invariance and Galilean transform

$$S_{\mathbb{R}}(e^{it\Delta} D(h_n^j) P(b_n^j) r_n^{j,L}) = S_{\mathbb{R}}(e^{it\Delta} r_n^{j,L}).$$

Let $r_n := \sum_{j=1}^J D(h_n^j)P(b_n^j)r_n^{j,L} + q_n$. By Lemma B.5 (B.10) and (B.21), we have

$$(B.23) \quad u_n = \sum_{j=1}^J f_n^j + q_n = \sum_{j=1}^J \sum_{l=1}^L \mathcal{G}_n^{j,l} \phi^{j,l} + r_n,$$

where $\mathcal{G}_n^{j,l} := D(h_n^j)P(b_n^j)U(s_n^{j,l})T(a_n^{j,l})$. We renumber $(j, l) \in \{1 \leq j \leq J, 1 \leq l \leq L\}$ as $k = 1, 2, \dots, K$ and rewrite (B.23) as

$$(B.24) \quad u_n = \sum_{k=1}^K \mathcal{G}_n^k \phi^k + r_n,$$

Note that $\{\mathcal{G}_n^k\}_n$ ($k = 1, 2, \dots, K$) are mutually orthogonal families of deformations. Indeed, if $k_1 \neq k_2$ we have either $j(k_1) \neq j(k_2)$, or $j(k_1) = j(k_2)$ and $l(k_1) \neq l(k_2)$, where $j(k)$ and $l(k)$ are numbers given by the above renumbering procedure, $k = (j, l)$. In the first case, the orthogonality follows from Lemma B.5. In the second case, Lemma B.6 gives the orthogonality, since $h_n^{k_1} \equiv h_n^{k_2}$ and $b_n^{k_1} \equiv b_n^{k_2}$ in this case.

By (B.22), (B.11), and Proposition 2.1, we have

$$S_{\mathbb{R}}(e^{it\Delta}u_n) \leq S_{\mathbb{R}}(e^{it\Delta}(u_n - r_n)) + C\varepsilon.$$

By assumption and Proposition 2.1,

$$S_{\mathbb{R}}(e^{it\Delta}(u_n - r_n)) \leq CM.$$

Combining the above inequality and Lemma B.7, one can verify that

$$S_{\mathbb{R}}(e^{it\Delta}u_n)^{(d+2)\alpha} \leq \sum_{k=1}^K S_{\mathbb{R}}(e^{it\Delta}\mathcal{G}_n^k \phi^k)^{(d+2)\alpha} + C\varepsilon + o(1)$$

as $n \rightarrow \infty$. Notice that $S_{\mathbb{R}}(e^{it\Delta}\mathcal{G}_n^k \phi^k) = S_{\mathbb{R}}(e^{it\Delta}\phi^k)$. By (B.8), we can take $\varepsilon = \varepsilon(m, M)$ small and n large enough to get

$$C_\alpha m^{(d+2)\alpha} \leq \sum_{k=1}^K S_{\mathbb{R}}(e^{it\Delta}\phi^k)^{(d+2)\alpha}.$$

On the other hand, by Proposition 2.1,

$$S_{\mathbb{R}}(e^{it\Delta}\phi^k) \leq C \left\| \phi^k \right\|_{\hat{M}_{2,r}^{d\alpha}}.$$

Since $(d+2)\alpha \geq \tilde{r} > r$, we have

$$\begin{aligned} C_\alpha m^{(d+2)\alpha} &\leq C \left(\sup_{1 \leq k \leq K} S_{\mathbb{R}}(e^{it\Delta}\phi^k) \right)^{(d+2)\alpha-r} \sum_{k=1}^K \left\| \phi^k \right\|_{\hat{M}_{2,r}^{d\alpha}}^r \\ &\leq C \left(\sup_{1 \leq k \leq K} S_{\mathbb{R}}(e^{it\Delta}\phi^k) \right)^{(d+2)\alpha-r} M^r. \end{aligned}$$

Thus, there exists k_0 such that

$$(B.25) \quad S_{\mathbb{R}}(e^{it\Delta}\phi^{k_0}) \geq C_\alpha \left(\frac{m^{(d+2)\alpha}}{M^r} \right)^{\frac{1}{(d+2)\alpha-r}}.$$

Now, up to subsequence, we have

$$(\mathcal{G}_n^{k_0})^{-1}u_n \rightharpoonup \phi^{k_0} + q_0 =: \psi \quad \text{weakly-}^* \text{ in } \hat{M}_{2,r}^{d\alpha}$$

as $n \rightarrow \infty$, where q_0 is a weak- $*$ limit of $(\mathcal{G}_n^{k_0})^{-1}q_n$. Indeed, by Lemma B.5 (B.10), we have

$$u_n = \sum_{1 \leq j \leq J, j \neq j_0} f_n^j + f_n^{j_0} + q_n,$$

where (j_0, l_0) is a pair corresponds to k_0 . By orthogonality obtained in Lemma B.5, one has $(\mathcal{G}_n^{j_0, l_0})^{-1}f_n^j \rightharpoonup 0$ weakly- $*$ in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$ for $j \neq j_0$. Further, $(\mathcal{G}_n^{j_0, l_0})^{-1}f_n^{j_0} \rightharpoonup \phi^{j_0, l_0}$ weakly- $*$ in $\hat{M}_{2,r}^{d\alpha}$ as $n \rightarrow \infty$. Therefore, we have the above limit and so $\psi \in \mathcal{M}(u)$.

Let us estimate $\|\psi\|_{\hat{M}_{2,r}^{d\alpha}}$ from below. Since $(\mathcal{G}_n^{k_0})^{-1}q_n$ is also bounded in $\hat{M}_{\tilde{q}, \tilde{r}}^{d\alpha}$ in light of (B.9), it converges to q_0 also weakly- $*$ in $\hat{M}_{\tilde{q}, \tilde{r}}^{d\alpha}$ as $n \rightarrow \infty$. One then sees from lower semi-continuity that

$$\|q_0\|_{\hat{M}_{\tilde{q}, \tilde{r}}^{d\alpha}} \leq \limsup_{n \rightarrow \infty} \|q_n\|_{\hat{M}_{\tilde{q}, \tilde{r}}^{d\alpha}} \leq C\varepsilon.$$

Thanks to (B.9), we have

$$S_{\mathbb{R}}(e^{it\Delta}q_0) \leq \|q_0\|_{\hat{M}_{\tilde{q}, \tilde{r}}^{d\alpha}} \leq C\varepsilon.$$

Finally, using Proposition 2.1 and (B.25), and choosing $\varepsilon = \varepsilon(m, M) > 0$ even smaller if necessary, we reach to the estimate

$$\begin{aligned} \|\psi\|_{\hat{M}_{2,r}^{d\alpha}} &\geq C(S_{\mathbb{R}}(e^{it\Delta}\mathcal{G}_n^{k_0}\phi^{k_0}) - S_{\mathbb{R}}(e^{it\Delta}\mathcal{G}_n^{k_0}q_0)) \\ &\geq CS_{\mathbb{R}}(e^{it\Delta}\phi^{k_0}) - C\varepsilon \\ &\geq \frac{C}{2} \left(\frac{m^{(d+2)\alpha}}{M^r} \right)^{\frac{1}{(d+2)\alpha-r}} =: \beta(m, M), \end{aligned}$$

which completes the proof of Theorem B.3. \square

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